This is kind of long. You’ll thank me later...

1. If $A$ is $m \times n$, and $m < n$, show that $A$ cannot have a left inverse. Similarly, if $A$ is $m \times n$, and $m > n$, show that $A$ cannot have a right inverse.

2. Let $M$ be an $n \times n$ matrix of integers. Assume $\det(M) \neq 0$, so $M$ is invertible (when viewed as a matrix of real, or complex, or rational numbers). Show that $M^{-1}$ is itself a matrix of integers if and only if $\det(M) = \pm 1$.

3. Let $S_n^{n \times n}$ be defined as

$$S_n^{n \times n} := \left\{ A \in \mathbb{R}^{n \times n} : A = A^T \right\}$$

(a) Show that $S_n^{n \times n}$ (along with the field $\mathbb{R}$) is a vector space.

(b) Find a basis for the space.

(c) What is the dimension of the space?

4. Let

$$A = \begin{bmatrix}
1 & 0 & -1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 \\
-1 & 1 & 0 & 1 & 1
\end{bmatrix}$$

Find a basis for $\{ x \in \mathbb{R}^5 : Ax = 0 \}$

5. Suppose $A \in \mathbb{C}^{n \times n}$. Show that $A$ is invertible if and only if $x = 0_{n \times 1}$ is the unique solution to $Ax = 0_{n \times 1}$.

6. Suppose $(V, F)$ is a vector space, $v_1, v_2 \in V$, and the set $\{v_1, v_2\}$ is a linearly independent set. For some $a, b, c, d \in F$, define $w_1 := av_1 + bv_2, w_2 := cv_1 + dv_2$. Show that $\{w_1, w_2\}$ is a linearly independent set if and only if $ad \neq bc$.

7. Let $(V, F)$ be a finite dimensional space. Suppose $\{v_1, v_2, \ldots, v_n\}$ be a basis for $V$. Let $A \in F^{n \times n}$. As usual, let $a_{ij}$ denote the $(i, j)$ element. Define vectors $\{u_i\}_{i=1}^n$ by the relation

$$u_i := \sum_{j=1}^n a_{ij}v_j$$

Show that $\{u_1, u_2, \ldots, u_n\}$ is a basis for $V$ if and only if $A$ is invertible.

8. Let $V$ be the set of polynomials with real coefficients of degree at most 3.

(a) Convince yourself that this is a vector space (over the field $\mathbb{R}$)

(b) Using $t$ as the indeterminate variable, show that $\{1, t, t^2, t^3\}$ is a basis for $V$.

(c) Show that $\{1, t - t^2, t + 2t^2, t^3 - t + 1\}$ is a basis for $V$. 

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(d) Define a map $\mathcal{A}$, mapping $V$ to $V$ by the rule

$$\mathcal{A}(v) := \frac{d}{dt} (tv)$$

Show that the map is a linear map.

(e) Find the matrix representation of $\mathcal{A}$ with respect to the basis choice in part (8b).

(f) Find the matrix representation of $\mathcal{A}$ with respect to the basis choice in part (8c).

(g) Calculate the determinants of both representations of $\mathcal{A}$.

9. Using the change-of-basis formula, show that if $\mathcal{A} \in \mathcal{L}(V, V)$, where $V$ is a finite dimensional vector space, then for any two matrix representations of $\mathcal{A}$, denoted $A_1$ and $A_2$, the determinants satisfy

$$\det A_1 = \det A_2$$

10. Let $A, B, C$ be matrices of appropriate size. In each case, assume that $X$ is restricted to be a matrix of appropriate dimension so that the expression is valid. Which of the following maps are linear?

(a) $f(X) := AX + XB$

(b) $f(X) := AX + BXC$

(c) $f(X) := AX + XBX$

(d) $f(X) := A'XA - X$

(e) $f(X) := \text{tr}(AX)$

11. Let $(V, F)$ be a finite dimensional vector space, and $\mathcal{A} \in \mathcal{L}(V, V)$. Assume that $\mathcal{A}$ is invertible, and denote its inverse as $\mathcal{A}^{-1}$. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for $V$. Associated with any $\mathcal{B} \in \mathcal{L}(V, V)$, use $M(\mathcal{B})$ to denote the matrix representation of $\mathcal{B}$ in this basis. Show that (as matrices)

$$[M(\mathcal{A})]^{-1} = M(\mathcal{A}^{-1})$$

12. $(V, F)$ is a vector space, and $R, T, S \subset V$ are subspaces. If $S \subset R$, show that

$$R \cap (S + T) (R \cap S) + (R \cap T)$$

13. Suppose $(V, F)$ and $(W, F)$ are vector spaces, and $\mathcal{A} \in \mathcal{L}(V, W)$. $S \subset W$ is a subspace. Define the inverse image of $S$ by $\mathcal{A}$ as

$$\{v \in V : \mathcal{A}(v) \in S\} \subset V$$

This is denoted $\mathcal{A}^{-1}(S)$, but is not to be confused with the inverse of $\mathcal{A}$ (since $\mathcal{A}$ may not be invertible).
(a) Show that this is a subspace of $V$

(b) If $V$ and $W$ are finite dimensional, show

$$\dim \left( A^{-1}(S) \right) = \dim (\ker A) + \dim (S \cap \text{Range } A)$$