1. Suppose $v \in \mathbb{C}^n$. Show that
\[
\|v\|_\infty \leq \|v\|_2 \leq \|v\|_1
\]
Also, find a vector $v$ such that all of the norms are equal, showing that the inequalities are “tight”.

2. Let $A \in \mathbb{C}^{n \times n}$. For $k = 1, 2, \ldots$, define $\mathcal{N}_k := \text{NullSpace} \left( A^k \right)$.
   (a) Show that for all $k$, $\mathcal{N}_k \subset \mathcal{N}_{k+1}$.
   (b) If for some integer $p$, $\mathcal{N}_p = \mathcal{N}_{p+1}$, then $\mathcal{N}_p = \mathcal{N}_q$ for all $q \geq p$.
   (c) Do the above results hold for general linear operators $A \in \mathcal{L}(V, V)$?

3. Consider a linear operator $A \in \mathcal{B}(\mathbb{C}^5, \mathbb{C}^4)$ defined by matrix multiplication: for $v \in \mathbb{C}^5$, $A(v) = Av$, where $A$ is
\[
A = \begin{bmatrix}
1 & 2 & 1 & 1 & -1 \\
-1 & -2 & 1 & 2 & 1 \\
1 & 1 & 3 & 4 & 1 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]
   (a) Find basis sets for $\text{Range}(A)$, $\text{Range}(A \circ A^*)$, $\text{Ker}(A^*)$, $\text{Ker}(A^* \circ A)$. Note these are all subspaces of $\mathbb{C}^4$.
   (b) Find basis sets for $\text{Range}(A^*)$, $\text{Range}(A^* \circ A)$, $\text{Ker}(A)$, $\text{Ker}(A \circ A^*)$. Note these are all subspaces of $\mathbb{C}^5$.
   (c) Verify that all of the orthogonality and direct sum properties hold among these subspaces.
   (d) With respect to the basis you chose for $\text{Range}(A)$ and $\text{Range}(A^*)$, find the matrix representation of the operator
\[
A|_{\text{Range}(A^*)} \to \text{Range}(A)
\]
Verify that this is an invertible operator (recall that it always is).

4. Using the data from problem 3, find the orthogonal projection onto $\text{Range}(A)$.

5. Using the data from problem 3, characterize and solve the least squares problem for $Ax = b$ with
\[
b = \begin{bmatrix}
1 \\
-4 \\
-3 \\
0
\end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}
\]
6. A particle of mass \( m(=2) \) is lying on a frictionless table. We apply a force \( u(t) \) at time \( t \). Suppose at \( t = 0 \), the particle is at rest, and (with respect to some reference on the table) the position is 0. Let \( d \) and \( v \) be the position and velocity of the particle.

(a) Show that for all \( t \geq 0 \)

\[
\begin{bmatrix}
  d(t) \\
  v(t)
\end{bmatrix} = \int_0^t \begin{bmatrix}
  \frac{1}{m} (t - \tau) \\
  \frac{1}{m}
\end{bmatrix} u(\tau) d\tau
\]

(b) Suppose we want to choose \( u \) so that at \( t = 1 \), the position and velocity satisfy \( d(1) = 2, v(1) = -1 \). Find the minimum norm force \( u \) that achieves this “transfer”, with

\[
\|u\|^2 = \int_0^1 u^2(t) dt
\]

Plot the force, displacement and velocity as functions of time.

(c) How would you modify your approach if the initial position and velocity (at \( t = 0 \)) were nonzero? What would the operator and its adjoint be?

7. Find a matrix \( A \in \mathbb{R}^{2 \times 2} \), with positive, real eigenvalues, such that for some nonzero vector \( x \), \( x^T Ax < 0 \). Can such an \( A \) be symmetric?

8. Suppose \( F \) is either \( \mathbb{R} \) or \( \mathbb{C} \). Using the matrix inversion lemma, and Schur complements, show the following: Given \( X \in F^{n \times n}, Y \in F^{n \times n} \), with \( X = X^* > 0, Y = Y^* > 0 \), and a positive integer \( r \). Show that there exist matrices \( X_2 \in F^{n \times r}, X_3 \in F^{r \times r} \) such that \( X_3 = X_3^* \) and

\[
\begin{bmatrix}
  X & X_2 \\
  X_2^* & X_3
\end{bmatrix} > 0 ,
\begin{bmatrix}
  X & X_2 \\
  X_2^* & X_3
\end{bmatrix}^{-1} = \begin{bmatrix}
  Y & ? \\
  ? & ?
\end{bmatrix}
\]

if and only if

\[
\begin{bmatrix}
  X & I_n \\
  I_n & Y
\end{bmatrix} \geq 0 ,\quad \text{rank} \left( X - Y^{-1} \right) \leq r.
\]

9. Suppose \( A \in \mathbb{C}^{n \times n} \) is Hermitian and positive definite, so \( A = A^* > 0 \). Show that

\[
\langle x, y \rangle_A := x^* Ay
\]

is an inner product on \( \mathbb{C}^n \). How/Why is the assumption of Hermitian important? How/Why is the assumption of positive-definiteness important?

10. The following is a singular value decomposition of \( A \in \mathbb{R}^{3 \times 2} \)

\[
A = \begin{bmatrix}
  3 & -4 & 0 \\
  1 & 0 & 0.1 \\
  0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
  1 & -\frac{\sqrt{3}}{2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  -\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{bmatrix}
\]
(a) Is $A^T A$ an invertible matrix? Is $AA^T$ an invertible matrix?
(b) What is an eigenvalue/eigenvector decomposition of $AA^T$?
(c) What is an orthonormal basis for the null space of $A^T$?
(d) What is an orthonormal basis for the range space of $A$?
(e) What is an orthonormal basis for the null space of $A$?
(f) What is $\|A\|_{2,2}$

11. Suppose $X, Y \in \mathbb{C}^{n \times n}$, with $X = X^*>0$, and $Y = Y^* \geq 0$. Show that

$$\max_{x \in \mathbb{C}^n \atop \|x\|_2 = 1} \frac{x^* Y x}{x^* X^{-1} x} = \max_{x \in \mathbb{C}^n \atop x \neq 0} \frac{x^* Y x}{x^* X^{-1} x} = \lambda_{\max} \left( X^{1/2} Y X^{1/2} \right) = \rho (XY)$$

What if $\mathbb{C}$ is replaced by $\mathbb{R}$, everywhere?

12. Suppose that $W \in \mathbb{C}^{n \times n}$, with $W = W^*$, and $L \in \mathbb{C}^{n \times n}$ is invertible. Show that $W < 0$ if and only if $L^* W L < 0$. Does the result hold when $<$ is replaced by one of $>, \leq, \geq$?

13. Find, by hand calculation the eigenvalues and the eigenvectors for the matrices

(a) $A = \begin{bmatrix} 5 & -4 \\ 12 & -9 \end{bmatrix}$

(b) $A = \begin{bmatrix} 14 & -8 \\ 24 & -14 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 1 \\ -9 & -4.8 \end{bmatrix}$

(d) $A = \begin{bmatrix} -2.4 & 1.8 \\ -1.8 & -2.4 \end{bmatrix}$