1. (a) Multiplying the differential equation by the integrating factor $e^{-\alpha \tau}$ and rearranging, we get

$$e^{-\alpha \tau} \dot{x}(\tau) - a e^{-\alpha \tau} x(\tau) = e^{-\alpha \tau} u(\tau).$$

Noticing that the left hand side is $\frac{d}{d\tau} e^{-\alpha \tau} x(\tau)$, we can integrate to get

$$\left[e^{-\alpha \tau} x(\tau)\right]_{\tau=0}^{t} = \int_{0}^{t} e^{-\alpha \tau} u(\tau) \, d\tau,$$

$$e^{-\alpha \tau} x(t) - x(0) = \int_{0}^{t} e^{-\alpha \tau} u(\tau) \, d\tau,$$

$$x(t) = e^{\alpha t} x_0 + \int_{0}^{t} e^{\alpha (t-\tau)} u(\tau) \, d\tau.$$  

(b) Since $|\cos \theta + j \sin \theta| = 1$ for all $\theta \in \mathbb{R}$,

$$|e^{\gamma t}| = \left|e^{\text{Re}(\gamma)t}\right| |\cos \text{Im}(\gamma)t + j \sin \text{Im}(\gamma)t| = e^{\text{Re}(\gamma)t}.$$

(c) We find points where the derivative of $te^{\alpha t}$ is zero.

$$\frac{d}{dt} te^{\alpha t} = 0$$

$$e^{\alpha t} + ate^{\alpha t} = 0$$

$$1 + at = 0$$

$$t = -\frac{1}{a} = \frac{1}{|a|}.$$

For this value of $t$, we have $te^{\alpha t} = \frac{1}{|a|}$. It can be checked that this is a maximum value by checking the signs of the derivative.

(d) Applying the previous question to $\frac{\alpha}{2}$, we find that $te^{\alpha t} \leq \frac{2}{|a|}e^{\alpha t}$ for all $t \geq 0$. Then,

$$te^{\alpha t} = te^{\frac{\alpha}{2}t}e^{\frac{\alpha}{2}t} \leq \frac{2}{|a|}e^{\frac{\alpha}{2}t}$$

for all $t \geq 0$. So $M = \frac{2}{|a|}$ and $\alpha = \frac{a}{2}$ are the desired constants.

(e) Assuming $\text{Re}(a) \neq \alpha$,

$$|\xi(t)| = \left|\int_{0}^{t} e^{\alpha(t-\tau)} w(\tau) \, d\tau\right|$$

$$\leq \int_{0}^{t} \left|e^{\alpha(t-\tau)}\right| \left|w(\tau)\right| \, d\tau$$

$$\leq \int_{0}^{t} e^{\text{Re}(a)(t-\tau)} Me^{\alpha \tau} \, d\tau$$

$$= Me^{\text{Re}(a)t} \int_{0}^{t} e^{-(\text{Re}(a)-\alpha)\tau} \, d\tau$$

$$= \frac{M}{\text{Re}(a)-\alpha} \left[e^{\text{Re}(a)t} - e^{\alpha t}\right].$$
If \( \text{Re}(a) < \alpha \), then we can set \( M_2 = \frac{M}{\text{Re}(a) - \alpha}, \) \( \alpha_2 = \text{Re}(a) < 0. \)

If \( \text{Re}(a) > \alpha \), then we can set \( M_2 = \frac{M}{\alpha - \text{Re}(a)}, \) \( \alpha_2 = \alpha < 0. \)

Finally, if \( \text{Re}(a) = \alpha \), we find that \( |\xi(t)| \leq M t e^{\text{Re}(a)t}. \) Then, by the result of the previous part, we can set \( M_2 = \frac{2M}{|\text{Re}(a)|} \), \( \alpha_2 = \frac{\text{Re}(a)}{2} < 0. \)

(f) Let the Schur decomposition of \( A \) be \( A = Q \Lambda Q^* \), where \( Q \) is unitary, and \( \Lambda \) is upper triangular. Then we define a new state vector \( z(t) = Q^* x(t) \), so that \( \dot{z}(t) = \Lambda z(t). \) Clearly, \( \lim_{t \to \infty} x(t) = 0_{n \times 1} \) if and only if \( \lim_{t \to \infty} z(t) = 0_{n \times 1}. \)

We say \( \lambda_j \) is the \( (i, j) \)th element of \( \Lambda \), so that \( \lambda_{ij} = 0 \) for \( i > j \), and \( \{\lambda_i\}_{i=1}^n \) are the eigenvalues of \( A \). We will prove the result inductively. We start by looking at the last element \( z_n(t) \) of the state vector \( z(t) \). Looking at the last row of the equation \( \dot{z}(t) = \Lambda z(t) \), we see that

\[
\dot{z}_n(t) = \lambda_{nn} z_n(t).
\]

By the results of parts (a) and (b),

\[
|z_n(t)| = |e^{\lambda_{nn} t} z_n(0)| = e^{\text{Re}(\lambda_{nn}) t} |z_n(0)|.
\]

Therefore, the state \( z_n(t) \) exponentially decays to zero for all initial conditions if and only if \( \text{Re}(\lambda_{nn}) < 0. \)

Now, suppose that we have shown that the each of the last \( (n-k) \) elements, \( z_{k+1}(t), z_{k+2}(t), \ldots, z_n(t) \), of the state vector \( z(t) \) exponentially decays to zero for all initial conditions. Now we look at the \( k \)th element, \( z_k(t) \). Looking at the \( k \)th row of the equation \( \dot{z}(t) = \Lambda z(t) \), we see that

\[
\dot{z}_k(t) = \lambda_{kk} z_k(t) + w(t),
\]

where

\[
w(t) = \sum_{j=k+1}^n \lambda_{kj} z_j(t).
\]

By the result of part (a),

\[
|z_k(t)| = \left| e^{\lambda_{kk} t} z_k(0) + \int_0^t e^{\lambda_{kk} (t-\tau)} w(t) \, d\tau \right| \\
\leq \left| e^{\lambda_{kk} t} z_k(0) \right| + \int_0^t e^{\lambda_{kk} (t-\tau)} |w(t)| \, d\tau.
\]

By part (b), we know that the the first term exponentially decays to zero if and only if \( \text{Re}(\lambda_{kk}) < 0. \) Also, \( w(t) \) exponentially decays to zero, so by part (e), we know that the the second term exponentially decays to zero if \( \text{Re}(\lambda_{kk}) < 0. \) Therefore, \( z_k(t) \) exponentially decays to zero for all initial conditions if and only if \( \text{Re}(\lambda_{kk}) < 0. \)

By induction, it can be seen that \( \lim_{t \to \infty} z(t) = 0_{n \times 1} \) if and only if \( \text{Re}(\lambda_{kk}) < 0 \) for all \( k. \) Therefore, the ODE is stable if and only if all eigenvalues of \( A \) have real parts less than 0.

(g) In terms of the new state vector \( z(t) \), the ODE becomes \( \dot{z}(t) = \Lambda z(t) + Q^* u. \) We can then repeat the analysis carried out in the previous part. Suppose that we have shown that \( z_{k+1}(t), z_{k+2}(t), \ldots, z_n(t) \) exponentially decay to zero for all initial conditions. Now the \( k \)th row of the equation is

\[
\dot{z}_k(t) = \lambda_{kk} z_k(t) + w_1(t) + w_2(t),
\]
where

\[ w_1(t) = \sum_{j=k+1}^{n} \lambda_{kj} z_j(t), \]

\[ w_2(t) = \sum_{i=1}^{n} q_{ik} u_i(t). \]

By part (a), since \( z_k(0) = 0 \),

\[
|z_k(t)| = \left| \int_0^t e^{\lambda_{kk}(t-\tau)} (w_1(t) + w_2(t)) \, d\tau \right|
\leq \left| \int_0^t e^{\lambda_{kk}(t-\tau)} w_1(t) \, d\tau \right| + \left| \int_0^t e^{\lambda_{kk}(t-\tau)} w_2(t) \, d\tau \right|.
\]

We are assuming that the ODE is stable without forcing, so \( \text{Re}(\lambda_{kk}) < 0 \). Now \( w_1(t) \) exponentially decays to zero, so by part (e), we know that the first term exponentially decays to zero. Also, since each \( u_i(t) \) exponentially decays to zero, \( w_2 \) also exponentially decays to zero, and again by part (e), we know that the second term exponentially decays to zero. Therefore, \( z_k(t) \) also exponentially decays to zero. By induction, this is true for all \( k \). Therefore, for some constants \( M_x, \alpha_x < 0 \), the solution satisfies

\[
\|x(t)\|_\infty \leq M_x e^{\alpha_x t}.
\]

(h) The result still holds when the initial conditions are nonzero. The analysis of the previous part holds, except we now have

\[
|z_k(t)| = \left| e^{\lambda_{kk}t} z_k(0) + \int_0^t e^{\lambda_{kk}(t-\tau)} (w_1(t) + w_2(t)) \, d\tau \right|
\leq \left| e^{\lambda_{kk}t} z_k(0) \right| + \left| \int_0^t e^{\lambda_{kk}(t-\tau)} w_1(t) \, d\tau \right| + \left| \int_0^t e^{\lambda_{kk}(t-\tau)} w_2(t) \, d\tau \right|.
\]

All three terms can be shown to exponentially decay to zero.

(i) Again, we repeat the same analysis, but now suppose that we have shown that \( z_{k+1}(t), z_{k+2}(t), \ldots, z_n(t) \) are bounded by some \( M_{w_1} \) (rather than exponentially decaying). Then \( w_1(t) \) is bounded. Furthermore, if \( \|u(t)\|_\infty \) is bounded by \( M_u \), then \( w_2(t) \) is bounded by \( M_{w_2} = \|Q^*\|_{\infty, \infty} M_u \). Then

\[
\left| \int_0^t e^{\lambda_{kk}(t-\tau)} w_1(t) \, d\tau \right| \leq \int_0^t |e^{\lambda_{kk}(t-\tau)} w_1(t)| \, d\tau
\leq \int_0^t e^{\text{Re}(\lambda_{kk})(t-\tau)} M_{w_1} \, d\tau
= \frac{M_{w_1}}{\text{Re}(\lambda_{kk})} \left[ 1 - e^{\text{Re}(\lambda_{kk})t} \right]
\leq \frac{M_{w_1}}{\text{Re}(\lambda_{kk})}.
\]

Again, we have

\[
|z_k(t)| \leq \left| \int_0^t e^{\lambda_{kk}(t-\tau)} w_1(t) \, d\tau \right| + \left| \int_0^t e^{\lambda_{kk}(t-\tau)} w_2(t) \, d\tau \right|.
\]

Both terms are bounded, so \( z_k(t) \) is also bounded. Then it is easy to show that \( \|z(t)\|_\infty \) is bounded, and then \( \|x(t)\|_\infty \) is bounded.
(j) The result still holds when the initial conditions are nonzero. The analysis of the previous part holds, except we now have

\[ |z_k(t)| \leq |e^{\lambda_k t}z_k(0)| + \left| \int_0^t e^{\lambda_k (t-\tau)}w_1(t) \, d\tau \right| + \left| \int_0^t e^{\lambda_k (t-\tau)}w_2(t) \, d\tau \right|. \]

The first term exponentially decays to zero, while the second and third terms are bounded, so \(z_k(t)\) is also bounded.

2. Let the SVD of \(A\) be

\[ A = [ \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \]

where \(\Sigma\) is \(k \times k\). So \(A = U_1 \Sigma V_1\).

Now we want to minimise

\[
\|Ax - b\|_2 = \|U_1 \Sigma V_1 x - b\|_2 \\
= \|U^*(U_1 \Sigma V_1 x - b)\|_2 \\
= \left\| \begin{bmatrix} U_1^* (U_1 \Sigma V_1 x - b) \\ U_2^* (U_1 \Sigma V_1 x - b) \end{bmatrix} \right\|_2 \\
= \left\| \begin{bmatrix} U_1^* U_1 - U_1^* b \\ -U_2^* b \end{bmatrix} \right\|_2, 
\]

since \(U_1^* U_1 = I_k\) and \(U_2^* U_1 = 0\).

Now, the last \((n-k)\) elements of this vector are \(-U_2^* b\), which is a constant. Thus, minimising the \(\| \cdot \|_2\) norm of this vector is equivalent to minimising \(\|\Sigma V_1 x - U_1^* b\|_2\). We find that this is equal to zero, for any \(x\) such that \(V_1 x = \Sigma^{-1} U_1^* b\). It can be shown that the set of all solutions to this equation is \(\{(V_1^* \Sigma^{-1} U_1^* b + V_2^* y) : y \in \mathbb{C}^{n-k}\}\). The least squares solution \(x\) with the minimum norm is then \(x = V_1^* \Sigma^{-1} U_1^* b\).

The same result can be verified by using the equations \(A^* A^* \xi = A^* b\), and \(x = A^* \xi\).

3. (a) \(A \preceq B\) and \(B \preceq A\).

\[ \Rightarrow u^*(A - B)u \leq 0 \text{ and } u^*(B - A)u \leq 0 \text{ for all } u \in \mathbb{C}^n. \]

\[ \Rightarrow u^*(A - B)u = 0 \text{ for all } u \in \mathbb{C}^n. \]

\[ \Rightarrow A = B. \]

\[ A \preceq B \text{ and } C \preceq D. \]

\[ \Rightarrow u^*(A - B)u \leq 0 \text{ and } u^*(C - D)u \leq 0 \text{ for all } u \in \mathbb{C}^n. \]

\[ \Rightarrow u^*((A + C) - (B + D))u \leq 0 \text{ for all } u \in \mathbb{C}^n. \]

\[ \Rightarrow A + C \preceq B + D. \]

(b) \(M \succ 0\).

\[ \Rightarrow u^* Mu \geq 0 \text{ for all } u \in \mathbb{C}^n. \]

\[ \Rightarrow (Lv)^* M (Lv) \geq 0 \text{ for all } v \in \mathbb{C}^n. \]

\[ \Rightarrow v^* (L^* M L) v \geq 0 \text{ for all } v \in \mathbb{C}^n. \]

\[ \Rightarrow L^* M L \succ 0. \]

\[ L^* M L \succ 0. \]

\[ \Rightarrow v^* (L^* M L) v \geq 0 \text{ for all } v \in \mathbb{C}^n. \]
\( (L^{-1}u)^*(L^*ML)(L^{-1}u) \geq 0 \) for all \( u \in \mathbb{C}^n \).
\( \Rightarrow u^*Mu \geq 0 \) for all \( u \in \mathbb{C}^n \).
\( \Rightarrow M \succ 0. \)

(c) \( u^*(W^*W)u = (Wu)^*(Wu) = \|Wu\|_2 \geq 0 \) for all \( u \in \mathbb{C}^n \).
\( \Rightarrow W^*W \succeq 0. \)

(d) \( u^*(W^*W)u = 0. \)
\( \Rightarrow \|Wu\|_2 = 0. \)
\( \Rightarrow Wu = 0. \)
\( \Rightarrow u = 0, \) since \( W \) has \( m \) linearly independent columns.

Therefore, \( W^*W \succ 0. \)

(e) Note that \( (M^{-1})^* = M^{-1}. \)
\( M \succeq 0. \)
\( \iff (M^{-1})^*MM^{-1} \succ 0, \) by part (b) with \( L = M^{-1}. \)
\( \iff M^{-1} \succ 0. \)

(f) The given matrix \( M \) has positive, real eigenvalues (both equal to 1). However,

\[
\begin{bmatrix}
  1 & -1 \\
  0 & 1
\end{bmatrix}
\begin{bmatrix}
  1 & 10 \\
  -1 & 1
\end{bmatrix}
= -8 < 0.
\]

So the theorem is not valid when \( M \) is not Hermitian.

4. In the case of real scalars, a matrix \( M = \begin{bmatrix} m \end{bmatrix} \) is positive definite (or positive semi-definite) if \( m \) is positive (or non-negative). The lemma then asserts that, given \( s > 0 \) and \( t \), the following inequality is true for all \( k, \)

\[
sk^2 - 2tk + \frac{t^2}{s} \geq 0,
\]

with equality when \( k = \frac{t}{s}. \) If we plot the left hand side of this inequality against \( k, \) the graph will be a convex parabola which touches the \( x \)-axis at \( k = \frac{t}{s}. \).