Basic Robustness Reduction to determinant condition

Given \( \Phi \subset C^{n \times n} \), with the property that
\[
\Delta \in \Phi, \quad 0 \leq \tau \leq 1 \Rightarrow \tau \Delta \in \Phi
\]
and \( M(s) \in S^{n \times n} \).

We want to determine if \( (I - M\Delta)^{-1} \in S^{n \times n} \) for all \( \Delta \in \Phi \).

This means that for \( G \in S^{n \times n} \), \( G^{-1} \in S^{n \times n} \) if and only if \( \det(G) \in \mathcal{U}_S \).

Hence, we simply need to check that
\[
\det(I - M\Delta) \in \mathcal{U}_S
\]
for all \( \Delta \in \Phi \).

This can be reduced to a (still complicated) condition that must be verified everywhere along the imaginary axes.

**Theorem 59** Given \( \Phi \) and \( M \) as above. Then \( \det(I - M\Delta) \in \mathcal{U}_S \) for all \( \Delta \in \Phi \) if and only if

1. \( \det(I - M(\infty)\Delta) \neq 0 \) for all \( \Delta \in \Phi \).
2. \( \det(I - M(j\omega)\Delta) \neq 0 \) for all \( \omega \in \mathbb{R} \), and all \( \Delta \in \Phi \).

**Proof:** If violated, use that \( \Delta \), giving a pole at \( \infty \) (so \( (I - M\Delta)^{-1} \) is improper) or on the imaginary axis.

On the other hand, if condition (which is along imaginary axes) is satisfied, we need to check it everywhere in the right-half-plane. Pick any \( \Delta \in \Phi \). Look at the function
\[
r_\tau(s) := \det(I - \tau M(s)\Delta) \in S
\]
Since \( \{M(j\omega) : \omega \in \mathbb{R}\} \cup M(\infty) \) is bounded, and \( M(\infty) \) exists (ie., \( M(j\omega) \) has a limit) for some \( 0 < \bar{\tau} < 1 \),
\[
|1 - r_\tau(j\omega)| \leq \frac{1}{2}
\]
for all $\omega$ (including $\infty$). Hence (by Nyquist) $r_{\bar{\tau}}$ has no zeros in RHP. Now, let $\tau$ change from $\bar{\tau}$ to 1. At no $\omega$ or any $\tau$ does it pass through zero (condition). Encirclements remain 0, hence $r_1(s)$ has no zeros in RHP (or at $s = \infty$). Therefore $r_1 \in U_S$ as desired. ♡.

This still holds if we let $\hat{\Phi}$ be dynamic. Given $\Phi$, define

$$\hat{\Phi} := \left\{ \hat{\Delta} \in S^{n \times n} : \Delta(\infty) \in \Phi, \forall \omega, \Delta(j\omega) \in \Phi \right\}.$$ 

Note that $\Phi \subset \hat{\Phi}$. Essentially the same proof gives:

**Theorem 60** Given $\Phi$ and $M$ as above, and $\hat{\Phi}$ defined from $\Phi$. Then $\det \left( I - M\hat{\Delta} \right) \in U_S$ for all $\hat{\Delta} \in \hat{\Phi}$ if and only if

1. $\det \left( I - M(\infty)\Delta \right) \neq 0$ for all $\Delta \in \Phi$.
2. $\det \left( I - M(j\omega)\Delta \right) \neq 0$ for all $\omega \in \mathbb{R}$, and all $\Delta \in \Phi$.

Now, go beyond robustness of stability. What about robustness of performance? Take $M \in S^{(n+n_e) \times (n+n_d)}$, and $\Phi$ as before. We want to determine if $(I - M_{11}\Delta)^{-1} \in S^{n \times n}$ for all $\Delta \in \Phi$, and if so, is

$$\sup_{\omega \in \mathbb{R}} \bar{\sigma} \left[ M_{22}(j\omega) + M_{21}(j\omega)\Delta \left( I - M_{11}(j\omega)\Delta \right)^{-1} M_{12}(j\omega) \right] < 1$$

for all $\Delta \in \Phi$.

This can be cast as a non-vanishing determinant condition as well.

**Theorem 61** Given $\Phi$, define

$$\Phi_P := \left\{ \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_2 \end{bmatrix} : \Delta \in \Phi, \Delta_2 \in C^{n_d \times n_e}, \bar{\sigma}(\Delta_2) \leq 1 \right\}$$

Then, the above question is true if and only if

$$\det \left( \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - M(j\omega)\Delta_P \right) \neq 0$$

for all $\Delta_P \in \Phi_P$ and $\omega \in \mathbb{R}$, including $\omega = \infty$. 

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**Proof:** If the determinant condition is violated, then at some \( \bar{\omega} \) (possibly \( \infty \)) there is a \( \bar{\Delta}_P \in \Phi_P \) that causes singularity. Recall that \( \bar{\Delta}_P \) is of the form

\[
\begin{bmatrix}
\bar{\Delta} & 0 \\
0 & \bar{\Delta}_2
\end{bmatrix}
\]

where \( \bar{\Delta} \in \Phi \) and where \( \bar{\Delta}_2 \in \mathbb{C}^{n_d \times n_c} \) with \( \bar{\sigma}(\bar{\Delta}_2) \leq 1 \). If \( I - M_{11}(\bar{\omega}) \bar{\Delta} \) is singular, then simply use it to create an unstable \( (I - M_{11}\Delta)^{-1} \). If \( I - M_{11}(\bar{\omega}) \bar{\Delta} \) is not singular, then recall from our determinant formulae that

\[
\det \left( \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - M(j\bar{\omega})\bar{\Delta}_P \right) = \det [I - M_{11}(j\bar{\omega})\bar{\Delta}] \det \left[ I - M_{22}(j\bar{\omega})\bar{\Delta}_2 - M_{21}(j\bar{\omega})\bar{\Delta}(I - M_{11}(j\bar{\omega})\bar{\Delta})^{-1} M_{12}(j\bar{\omega})\bar{\Delta}_2 \right]
\]

Since this is 0, the 2nd term must be zero (the first isn’t), and since \( \bar{\sigma}(\bar{\Delta}_2) \leq 1 \) it must be that

\[
\bar{\sigma} \left[ M_{22}(j\bar{\omega}) + M_{21}(j\bar{\omega})\bar{\Delta}(I - M_{11}(j\bar{\omega})\bar{\Delta})^{-1} M_{12}(j\bar{\omega}) \right] \geq 1
\]

(here we use that if \( I - AB \) is singular, and \( \bar{\sigma}(B) \leq 1 \), then it must be that \( \bar{\sigma}(A) \geq 1 \)).

The reverse argument is essentially the same, and you should do it. \( \sharp \).

By the exact same reasoning used earlier, the theorem claim can be extended to \( \hat{\Phi} \) while the checking of the non-vanishing determinant still takes place only on \( \Phi \).

**Conclusion:** We have reduced all of these questions to frequency-dependent checking of non-vanishing determinants of an expression \( I - M\Psi \) where \( M \) is fixed, and \( \Psi \) lives in a set. That new problem becomes the focus of our attention. It is referred to “structured singular value” theory, and involves much/most of the linear algebra and convexity ideas that we have learned.