1. Set notation
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3. More notation
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5. Matrix Facts (determinants, inversion formulae)
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Some Notation

1. $\mathbb{R}$ is the set of real numbers. $\mathbb{C}$ is the set of complex numbers.

2. $\mathbb{N}$ is the set of integers.

3. The set of all $n \times 1$ column vectors with real number entries is denoted $\mathbb{R}^n$. The $i$'th entry of a column vector $x$ is denoted $x_i$.

4. The set of all $n \times m$ rectangular matrices with complex number entries is denoted $\mathbb{C}^{n \times m}$. The element in the $i$'th row, $j$'th column of a matrix $M$ is denoted by $M_{ij}$, or $m_{ij}$.

5. Set notation:
   
   (a) $a \in A$ is read: “$a$ is an element of $A$”
   (b) $X \subset Y$ is read: “$X$ is a subset of $Y$”
   (c) If $A$ and $B$ are sets, then $A \times B$ is a new set, consisting of all ordered-pairs drawn from $A$ and $B$,

   $$A \times B := \{(a, b) : a \in A, b \in B\}$$

   (d) The expression $\{A : B\}$ is read as:

   “The set of all $\text{insert expression } A$ such that $\text{insert expression } B$.”

   Hence

   $$\left\{x \in \mathbb{R}^3 : \sum_{i=1}^{3} x_i^2 \leq 1\right\}$$

   is the ball of radius 1, centered at the origin, in 3-dimensional euclidean space.

6. The notation $f : X \to Y$ implies that $X$ and $Y$ are sets, and $f$ is a function mapping $X$ into $Y$.
Fields

A field consists of: a set $\mathcal{F}$ (which must contain at least 2 elements) and two operations, addition (+) and multiplication ($\cdot$), each mapping $\mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$. Several axioms must be satisfied:

- For every $a, b \in \mathcal{F}$, there corresponds an element $a + b \in \mathcal{F}$, the addition of $a$ and $b$. For all $a, b, c \in \mathcal{F}$, it must be that
  
  $a + b = b + a$
  
  $(a + b) + c = a + (b + c)$

- There is a unique element $\theta \in \mathcal{F}$ (or $0_{\mathcal{F}}$, $\theta_{\mathcal{F}}$, or just 0) such that for every $a \in \mathcal{F}$, $a + \theta = a$. Moreover, for every $a \in \mathcal{F}$, there is a unique element labeled $-a$ such that $a + (-a) = \theta$.

- For every $a, b \in \mathcal{F}$, there corresponds an element $a \cdot b \in \mathcal{F}$, the multiplication of $a$ and $b$. For every $a, b, c \in \mathcal{F}$
  
  $a \cdot b = b \cdot a$
  
  $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

- There is a unique element $1_{\mathcal{F}} \in \mathcal{F}$ (or just 1) such that for every $a \in \mathcal{F}$, $1 \cdot a = a \cdot 1 = a$. Moreover, for every $a \in \mathcal{F}$, $a \neq \theta$, there is a unique element, labeled $a^{-1} \in \mathcal{F}$ such that $a \cdot a^{-1} = 1_{\mathcal{F}}$.

- For every $a, b, c \in \mathcal{F}$,
  
  $a \cdot (b + c) = a \cdot b + a \cdot c$

Example: The real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, and the rational numbers $\mathbb{Q}$ are three examples of fields.
Vector Spaces

A vector space consists of:

• a set $\mathcal{V}$, whose elements are called “vectors,” and
• a field $\mathcal{F}$ (often just $\mathbb{R}$ or $\mathbb{C}$, and then denoted $\mathcal{F}$) whose elements are “scalars.”

Two operations,

• addition of vectors, and
• scalar multiplication

are defined and must satisfy the following relationships:

• For every $u, w \in \mathcal{V}$, there corresponds a vector $u + w \in \mathcal{V}$ such that for all $u, v, w \in \mathcal{V}$
  
  1. $u + w = w + u$
  2. $(u + w) + v = u + (w + v)$

  There is a unique vector $\theta_\mathcal{V}$ (or $0_\mathcal{V}$, $\theta$, or just 0) such that for every $w \in \mathcal{V}$, $w + \theta_\mathcal{V} = w$. Moreover, for every $w \in \mathcal{V}$, there is a unique vector labeled $-w$ such that $w + (-w) = \theta_\mathcal{V}$.

• For every $\alpha \in \mathcal{F}$ and $w \in \mathcal{V}$ there corresponds a vector $\alpha w \in \mathcal{V}$. The operation must satisfy $1w = w$ for all $w \in \mathcal{V}$ and for every $u, w \in \mathcal{V}, \alpha, \beta \in \mathcal{F}$ the distributive laws

  1. $\alpha(u + w) = \alpha u + \alpha w$
  2. $(\alpha + \beta)u = \alpha u + \beta u$

  must hold.
If $Z$ and $W$ are vector spaces over the same $\mathcal{F}$, then $Z \times W$ is also a vector space (field $\mathcal{F}$), with addition and scalar multiplication defined “coordinatewise.”

Specifically, if $q_1, q_2 \in Z \times W$, then each $q_i$ is of the form

$$q_i = (z_i, w_i).$$

For $\alpha \in \mathcal{F}$, define

$$\alpha q_1 := (\alpha z_1, \alpha w_1), \quad q_1 + q_2 := (z_1 + z_2, w_1 + w_2).$$
Vector Spaces

Simple Examples

• \( n > 0, \mathcal{V} = \mathbb{R}^n, \mathcal{F} = \mathbb{R} \), addition and scalar multiplication defined in terms of components

\[(x + y)_i := x_i + y_i, \quad (\alpha x)_i := \alpha x_i\]

• \( n > 0, \mathcal{V} = \mathbb{C}^n, \mathcal{F} = \mathbb{C} \), addition and scalar multiplication again defined in terms of components.

• \( n > 0, \mathcal{V} = \mathbb{C}^n, \mathcal{F} = \mathbb{R} \), addition and scalar multiplication again defined in terms of components.

• \( n, m > 0, \mathcal{V} = \mathbb{F}^{n \times m}, \mathcal{F} = \mathbb{F} \), addition and scalar multiplication defined entrywise

\[(A + B)_{i,j} := A_{i,j} + B_{i,j}, \quad (\alpha A)_{i,j} := \alpha A_{i,j}\]

• \( \mathcal{V} := \) all continuous, real-valued functions defined on \([0, 1]\), \( \mathcal{F} = \mathbb{R} \). Addition and scalar multiplication defined pointwise: for \( f, g \in \mathcal{V}, \alpha \in \mathbb{R} \)

\[(f + g)(x) := f(x) + g(x), \quad (\alpha f)(x) := \alpha f(x)\]

• \( \mathcal{V} := \) all piecewise continuous, real-valued functions defined on \([0, \infty)\), with a finite number of discontinuities in any finite interval, \( \mathcal{F} = \mathbb{R} \). Addition and scalar multiplication defined pointwise, as before. For future, call this space \( \text{PC}[0, \infty) \).

• Same function space as above, with further restriction that

\[
\max_{x \geq 0} |f(x)| < \infty \quad \text{or} \quad \int_0^\infty |f(\eta)| \, d\eta < \infty
\]

Call these \( \text{PC}_\infty[0, \infty) \), and \( \text{PC}_1[0, \infty) \), respectively.
1. In a statement, if \( \mathbf{F} \) appears, it means that the statement is true with \( \mathbf{F} \) replaced by either \( \mathbf{R} \) or \( \mathbf{C} \) throughout the statement.

2. The set of all \( n \times 1 \) column vectors with real number entries is denoted \( \mathbf{R}^n \).

3. The set of all \( n \times m \) rectangular matrices with complex number entries is denoted \( \mathbf{C}^{n \times m} \). The element in the \( i \)’th row, \( j \)’th column of a matrix \( M \) is denoted by \( M_{ij} \), or \( m_{ij} \).

4. If \( x \in \mathbf{C}, \bar{x} \in \mathbf{C} \) is the complex conjugate of \( x \).

5. If \( M \in \mathbf{F}^{n \times m} \), then \( M^T \) is the transpose of \( M \); \( M^* \) is the complex-conjugate transpose of \( M \)

6. If \( Q \in \mathbf{F}^{n \times n} \), and \( Q^*Q = I_n \), then \( Q \) is called unitary.

7. \( \mathbf{R}_+ := \{ \alpha \in \mathbf{R} : \alpha \geq 0 \} \), \( \mathbf{N}_+ := \{ k \in \mathbf{N} : k \geq 0 \} \)
1. Eigenvalues: $\lambda \in \mathbb{C}$ is an eigenvalue of $M \in \mathbb{F}^{n \times n}$ if there is a vector $v \in \mathbb{C}^n, v \neq 0_n$, such that
\[ Mv = \lambda v \]
The vector $v$ is called an eigenvector associated with eigenvalue $\lambda$.

2. The eigenvalues of $M \in \mathbb{F}^{n \times n}$ are the roots of the equation
\[ p_M(\lambda) := \text{det} (\lambda I_n - M) = 0 \]

3. Fact: Every matrix has at least one eigenvalue and associated eigenvector, since the polynomial $p_M(\lambda)$ has at least one root.

4. Fact: The eigenvalues of a matrix are continuous functions of the entries of the matrix.

5. For any $n \times m$ matrix $A$, and $m \times n$ matrix $B$, the nonzero eigenvalues of $AB$ are equal to the nonzero eigenvalues of $BA$.

6. A matrix $M \in \mathbb{F}^{n \times n}$ is called Hurwitz if all of its eigenvalues have negative real parts.

7. A matrix $M \in \mathbb{F}^{n \times n}$ is called Schur if all of its eigenvalues have absolute value less than 1.
Linear Algebra

Determinant Facts

1. If $A$ and $B$ are square matrices, then
   
   (a) $\det(AB) = \det(BA) = \det(A)\det(B)$
   
   (b) $\det(A) = \det(A^T)$
   
   (c) $\det(A^*) = \overline{\det(A)}$

2. For any $n \times m$ matrix $A$, and $m \times n$ matrix $B$,
   
   (a) $\det(I_n + AB) = \det(I_m + BA)$
   
   (b) $(I_n + AB)$ is invertible if and only if $(I_m + BA)$ is invertible, and moreover,
   
   (c) $(I_n + AB)^{-1} A = A (I_m + BA)^{-1}$

3. If $X$ and $Z$ are square, $Y$ compatible, then
   
   \[
   \det\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} = \det(X)\det(Z)
   \]

4. If $X$ and $Z$ are square, invertible, $Y$ compatible, then
   
   \[
   \begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}^{-1} = \begin{bmatrix} X^{-1} & 0 \\ -Z^{-1}YX^{-1} & Z^{-1} \end{bmatrix}
   \]

5. If $A$ and $D$ are square, $D$ invertible, $B, C$ compatible dimensions, then
   
   \[
   \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ C & D \end{bmatrix}
   \]
   
   so that
   
   \[
   \det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A - BD^{-1}C)\det(D)
   \]
1. Suppose $A$ and $D$ are square, $D$ invertible, $B, C$ compatible dimensions. If $A - BD^{-1}C$ is invertible then
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}^{-1} = \begin{bmatrix}
I & 0 \\
-D^{-1}C & D^{-1}
\end{bmatrix} \begin{bmatrix}
(A - BD^{-1}C)^{-1} & -I \\
0 & I
\end{bmatrix} \begin{bmatrix}
(A - BD^{-1}C)^{-1} - (A - BD^{-1}C)^{-1}BD^{-1} \\
0 & I
\end{bmatrix} = \begin{bmatrix}
(A - BD^{-1}C)^{-1} & -I \\
-D^{-1}C (A - BD^{-1}C)^{-1} & D^{-1}C (A - BD^{-1}C)^{-1} BD^{-1} + D^{-1}
\end{bmatrix}
\]

2. If $A$ and $D$ are square, invertible, $B, C$ compatible dimensions, then
\[
\det(D) \det \left( A - BD^{-1}C \right) = \det(A) \det \left( D - CA^{-1}B \right)
\]
and if not 0, then
\[
\left( A - BD^{-1}C \right)^{-1} = A^{-1} + A^{-1}B \left( D - CA^{-1}B \right)^{-1} CA^{-1}
\]

3. If $A$ is square and invertible, and $B, C$ and $D$ are compatibly dimensioned, then vectors $d_1, d_2, e_1$ and $e_2$ satisfy
\[
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix} = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \begin{bmatrix}
d_1 \\
d_2
\end{bmatrix}
\]
if and only if they satisfy
\[
\begin{bmatrix}
d_1 \\
e_2
\end{bmatrix} = \begin{bmatrix}
A^{-1} & -A^{-1}B \\
CA^{-1} & D - CA^{-1}B
\end{bmatrix} \begin{bmatrix}
e_1 \\
d_2
\end{bmatrix}
\]
In reparametrizing some optimization problems involving feedback, the following is useful: Let $T \in F^{m \times m}$ be given. Define

$$S_1 := \{K (I - TK)^{-1} : K \in F^{m \times n}, \det (I - TK) \neq 0\}$$

$$S_2 := \{Q \in F^{m \times n} : \det (I - QT) \neq 0\}$$

Then $S_1 = S_2$, and $S_2$ is dense in $F^{m \times n}$; that is, for any $\tilde{Q} \in F^{m \times n}$, and any $\epsilon > 0$, there is a $Q \in S_2$ such that

$$\max_{1 \leq i \leq m, 1 \leq j \leq n} |\tilde{q}_{ij} - q_{ij}| < \epsilon$$
Suppose \((V, F)\) is a vector space (again, \(F\) is either \(\mathbb{R}\) or \(\mathbb{C}\)). If there is a function \(\|\cdot\| : V \to \mathbb{R}\) such that for any \(u, v \in V\), and \(\alpha \in F\)

- \(\|u\| \geq 0\)
- \(\|u\| = 0 \iff u = 0_n\)
- \(\|\alpha u\| = |\alpha| \|u\|\)
- \(\|u + v\| \leq \|u\| + \|v\|\)

then the function \(\|\cdot\|\) is called a norm on \(V\), and \((V, F)\) is a normed vector space
For a vector $v \in \mathbf{F}^n$, let $v_i$ be the $i$'th component. Define

$$\|v\|_1 := \sum_{i=1}^{n} |v_i|$$

$$\|v\|_2 := \left( \sum_{i=1}^{n} |v_i|^2 \right)^{1/2}$$

$$\|v\|_{\infty} := \max_{1 \leq i \leq n} |v_i|$$

Each of these separate definitions satisfy all of the 4 axioms that a norm must satisfy (all axioms are easy to check except triangle inequality for $\|\cdot\|_2$, which we will verify in a few slides).

Hence each of $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_{\infty}$ are norms on $\mathbf{F}^n$.

We will pretty much exclusively use the $\|\cdot\|_2$ norm and often drop the subscript 2, simply using $\|\cdot\|$. Some easy facts are

1. For $v \in \mathbf{F}^n$, $\|v\|^2 = v^*v$

2. For $v \in \mathbf{F}^n, w \in \mathbf{F}^m$, $\left\| \begin{array}{c} v \\ w \end{array} \right\|^2 = \|v\|^2 + \|w\|^2$.

3. If $Q \in \mathbf{F}^{n \times n}$, $Q^*Q = I_n$, then for all $v \in \mathbf{F}^n$, $\|Qv\| = \|v\|$

4. Given $Q \in \mathbf{F}^{n \times n}$, $Q^*Q = I_n$,

$$\{ x : x \in \mathbf{F}^n, \|x\| \leq 1 \} = \{ Qx : x \in \mathbf{F}^n, \|x\| \leq 1 \}$$

and

$$\{ x : x \in \mathbf{F}^n, \|x\| = 1 \} = \{ Qx : x \in \mathbf{F}^n, \|x\| = 1 \}$$
Inner Product Spaces

A vector space \((\mathcal{V}, \mathbb{F})\) is an inner product space if there is a function \(<\cdot, \cdot>: \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}\) such that for every \(u, v, w \in \mathcal{V}\) and \(\alpha \in \mathbb{F}\) the following hold:

1. \(\langle u, v \rangle = \langle v, u \rangle\)
2. \(\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle\)
3. \(\langle \alpha u, w \rangle = \bar{\alpha} \langle u, w \rangle\)
4. \(\langle u, u \rangle \geq 0\)
5. \(\langle u, u \rangle = 0\) if and only if \(u = 0\).

The function \(\langle \cdot, \cdot \rangle\) is called the inner product on \(\mathcal{V}\).

Two vectors \(u, w \in \mathcal{V}\) are said to be perpendicular, written \(u \perp w\) if \(\langle u, w \rangle = 0\).

The most important inner product spaces that we will use in this section are \((\mathbb{R}^n, \mathbb{R})\) and \((\mathbb{C}^n, \mathbb{C})\), with inner products defined as

\[
\langle u, w \rangle = \sum_{i} u_i w_i = u^T w
\]

\[
\langle u, w \rangle = \sum_{i} \bar{u}_i w_i = u^* w
\]
On $(\mathcal{V}, F)$, define a function using by the inner-product. For each $v \in \mathcal{V}$ define

$$N(v) := \sqrt{\langle v, v \rangle}$$

The Schwarz inequality relates inner products and $N$.

**Theorem:** For each $u, w \in \mathcal{V}$ $|\langle u, w \rangle| \leq N(u)N(w)$.

**Proof:** Given $u$ and $w$, find complex number $\alpha$ with $|\alpha| = 1$, and $\alpha \langle u, w \rangle = |\langle u, w \rangle|$. Then for any real number $t$,

$$0 \leq \langle u + t\alpha w, u + t\alpha w \rangle = N(u)^2 + 2t |\langle u, w \rangle| + t^2 N(w)^2.$$  

This is a quadratic function. Characterizing that the minimum (over the real variable $t$) is non-negative gives the result.

$$|\langle u, w \rangle| \leq N(u)N(w)$$

The triangle inequality follows for $N$ as well: Given any $u, w \in \mathcal{V}$,

$$N(u + w)^2 = \langle u + w, u + w \rangle$$

$$= N(u)^2 + 2\text{Re} (\langle u, w \rangle) + N(w)^2$$

$$\leq N(u)^2 + 2 |\langle u, w \rangle| + N(w)^2$$

$$\leq N(u)^2 + 2N(u)N(w) + N(w)^2$$

$$= \left( N(u) + N(w) \right)^2$$

Hence, $N$ is actually a norm on $\mathcal{V}$, so every inner-product space is in fact a normed vector space, using $N$, the norm induced from the inner product. So, unless otherwise notated, using the symbol $\| \cdot \|$ when working with a inner-product space means the norm induced from the inner product.

Note, if $u$ and $w$ are perpindicular, then $\|u + w\|^2 = \|u\|^2 + \|w\|^2$, which is the “Pythagorean” theorem.
Take $A \in \mathbb{C}^{n \times m}$. Then

1. The $m$ columns of $\begin{bmatrix} I_m \\ A \end{bmatrix}$ are linearly independent, and are perpendicular to the $n$ linearly independent columns of $\begin{bmatrix} -A^* \\ I_n \end{bmatrix}$

2. Take $n > m$, and assume the columns of $A$ are linearly independent. Suppose $A_\perp$ is $n \times (n - m)$, has linearly independent columns, and $A^*_\perp A = 0$. If $X$ is $n \times n$, and invertible, then $XA$ and $X^{-*}A_\perp$ each have linearly independent columns, and are perpendicular to one another.
Linear Transformations on Vector Spaces

Suppose $\mathcal{V}$ and $\mathcal{W}$ are vector spaces over the same field $\mathcal{F}$. If $\mathcal{L}: \mathcal{V} \to \mathcal{W}$ satisfies

$$\mathcal{L}(\alpha v + \beta u) = \alpha \mathcal{L}(v) + \beta \mathcal{L}(u)$$

for all $\alpha, \beta \in \mathcal{F}$, and all $v, u \in \mathcal{V}$, then $\mathcal{L}$ is a linear transformation on $\mathcal{V}$ to $\mathcal{W}$.

Examples:

1. $\mathcal{V} = \mathbb{C}^m$, $\mathcal{W} = \mathbb{C}^n$, $M \in \mathbb{C}^{n \times m}$, and $\mathcal{L}$ defined by matrix-vector multiplication: For $v \in \mathcal{V}$, define $\mathcal{L}(v)$ as

$$\mathcal{L}(v) := Mv,$$

or componentwise $(\mathcal{L}(v))_i := \sum_{j=1}^{m} M_{ij}v_j$

2. $\mathcal{V} = \mathbb{R}^{n \times n}$, $\mathcal{W} = \mathbb{R}^{n \times n}$, $A \in \mathbb{R}^{n \times n}$, and $\mathcal{L}$ defined by a Lyapunov operator, For $P \in \mathcal{V}$, define $\mathcal{L}(P)$ as

$$\mathcal{L}(P) := A^T P + PA$$

3. $\mathcal{V} = PC_{\infty}[0, \infty)$, $\mathcal{W} = PC_{\infty}[0, \infty)$, $g \in PC_1[0, \infty)$, and $\mathcal{L}$ defined by convolution, For $v \in \mathcal{V}$, define $\mathcal{L}v$ as

$$(\mathcal{L}v)(t) := \int_0^t g(t - \tau)v(\tau)d\tau$$

For the remainder of this handout, focus on the linear operator defined by matrix-vector multiplication, and other results about matrices.
If $M \in \mathbb{F}^{n \times m}$, then $M$ naturally defines a linear transformation $L_M : \mathbb{F}^m \rightarrow \mathbb{F}^n$ via standard matrix-vector multiplication.

For any $v \in \mathbb{R}^m$

$$L_M(v) := Mv$$

Typically, we will not take care to distinguish the matrix from the operation. Simply note that matrix-vector multiplication in a linear transformation on the vector, namely, for all $u, v \in \mathbb{F}^m$, $\alpha, \beta \in \mathbb{F}$,

$$M (\alpha u + \beta v) = \alpha Mu + \beta Mv$$

Using norms in $\mathbb{F}^m$ and $\mathbb{F}^n$, the norm of the matrix transformation can be characterized

Define

$$\|M\|_{\alpha \rightarrow \beta} := \max_{u \in \mathbb{F}^m, u \neq 0} \frac{\|Mu\|_\alpha}{\|u\|_\beta}$$

This is the maximum amplification obtainable, via matrix-vector multiplication, measuring sizes in the domain and range with norms.
Easy Facts: For $M \in \mathbf{F}^{n \times m}$,

1. Other characterizations are possible

$$
\|M\|_{\alpha \leftarrow \beta} = \max_{u \in \mathbf{R}^m : \|u\|_\beta \leq 1} \|Mu\|_\alpha = \max_{u \in \mathbf{R}^m : \|u\|_\beta = 1} \|Mu\|_\alpha
$$

2. Easily proven: $\|M\|_{1 \leftarrow 1} = \max_{1 \leq j \leq m} \sum_{i=1}^{n} |M_{ij}|$

3. Easily proven: $\|M\|_{\infty \leftarrow \infty} = \max_{1 \leq i \leq n} \sum_{j=1}^{m} |M_{ij}|$

4. Later: $\|M\|_{2 \leftarrow 2}$ is characterized in terms of the eigenvalues of $M^*M$.

5. Interchanging rows and/or columns of $M$ does not change $\|M\|_{1 \leftarrow 1}$, $\|M\|_{2 \leftarrow 2}$, or $\|M\|_{\infty \leftarrow \infty}$.

6. Given $U \in \mathbf{F}^{n \times n}, V \in \mathbf{F}^{m \times m}$ both unitary (ie., $U^*U = I_n, V^*V = I_m$), then for any $M \in \mathbf{F}^{n \times m}$,

$$
\|UMV\|_{2 \leftarrow 2} = \|M\|_{2 \leftarrow 2}
$$

7. If $\|M\|_{\alpha \leftarrow \alpha} < 1$, then $\det (I - M) \neq 0$

8. For matrices $A, B, C$ of appropriate dimensions,

$$
\|AB\|_{\alpha \leftarrow \gamma} \leq \|A\|_{\alpha \leftarrow \beta} \|B\|_{\beta \leftarrow \gamma}
$$

$$
\|A + C\|_{\alpha \leftarrow \gamma} \leq \|A\|_{\alpha \leftarrow \gamma} + \|C\|_{\alpha \leftarrow \gamma}
$$

9. Deleting rows and/or columns does not increase $\|\cdot\|_{p \leftarrow p}$. Specifically, for matrices $A, B, C$ of appropriate dimensions,

$$
\left\| \begin{bmatrix} A & B \end{bmatrix} \right\|_{p \leftarrow p} \geq \|A\|_{p \leftarrow p}, \quad \left\| \begin{bmatrix} A \\ C \end{bmatrix} \right\|_{p \leftarrow p} \geq \|A\|_{p \leftarrow p}
$$
**Theorem:** Given a matrix \( A \in \mathbb{C}^{n \times n} \). There exists a matrix \( Q \in \mathbb{C}^{n \times n} \) with

- \( Q^*Q = I_n \), and
- \( Q^*AQ =: \Lambda \) upper triangular.

**Remarks:**

1. Proof is straightforward – induction along with Gram-Schmidt Orthonormalization process.

2. The matrix \( Q \) has orthonormal rows and columns (since \( Q^*Q = QQ^* = I_n \))

3. Since \( Q^*AQ \) is upper triangular, the eigenvalues of \( Q^*AQ \) are the diagonal entries.

4. In this case, \( Q^{-1} = Q^* \), so the eigenvalues of \( Q^*AQ \) are the same as the eigenvalues of \( A \). The order that the eigenvalues appear is arbitrary (they can be sorted in any order). This will be clear from the proof.

5. The Matlab command \texttt{schur} computes (reliably and quickly) a Schur decomposition.
Note that the theorem is true for \(1 \times 1\) matrices, ie., \(n = 1\), simply take \(Q := 1\), and \(\Lambda = A\).

Now, suppose that the theorem statement is true for \(n = k\), ie., suppose it is true for \(k \times k\) matrices. Furthermore, let \(A \in \mathbb{F}^{(k+1) \times (k+1)}\). Let \(v \in \mathbb{C}^{k+1}\) be an eigenvector of \(A\), with corresponding eigenvalue \(\lambda \in \mathbb{C}\) (possible since every matrix has at least one eigenvalue). By definition, \(v \neq 0_{k+1}\), and hence we can (by dividing) assume that \(v^*v = 1\). Now, using the Gram-Schmidt orthogonalization procedure, choose vectors \(v_1, v_2, \ldots, v_k\) each in \(\mathbb{C}^{k+1}\) such that

\[
\{v, v_1, v_2, \ldots, v_k\}
\]

is a set of mutually orthonormal vectors. Stack these into a square, \((k + 1) \times (k + 1)\) matrix \(V := [v \ v_1 \ v_2 \ \cdots \ v_k]\).

Note that \(V^*V = I_{k+1}\). Moreover, there is a matrix \(\Gamma \in \mathbb{C}^{k \times k}\), and a vector \(w \in \mathbb{C}^k\) such that

\[
AV = V \begin{bmatrix} \lambda & w^* \\ 0 & \Gamma \end{bmatrix}
\]

By then induction hypothesis, since \(\Gamma\) is of dimension \(k\), there is a matrix \(P \in \mathbb{C}^{k \times k}\) and upper triangular \(\Psi \in \mathbb{C}^{k \times k}\) with \(P^*P = I_k\) and \(P^*\Gamma P = \Psi\). Hence, we have

\[
\begin{bmatrix} 1 & 0 \\ 0 & P^* \end{bmatrix} V^*AV \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & P^* \end{bmatrix} \begin{bmatrix} \lambda & w^* \\ 0 & \Gamma \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix} = \begin{bmatrix} \lambda & w^*P \\ 0 & \Psi \end{bmatrix}
\]

which is indeed upper triangular. Moreover

\[
Q := V \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix}
\]

has \(Q^*Q = I_{k+1}\) as desired. \(\sharp\)
**Linear Algebra**  
**Symmetric, Hermitian, Normal Matrices**

**Definition:** The set of real, symmetric $n \times n$ matrices is denoted $\mathcal{S}^{n\times n}$, and defined as

$$\mathcal{S}^{n\times n} := \{ M \in \mathbb{R}^{n\times n} : M^T = M \}$$

**Definition:** The set of complex, Hermitian $n \times n$ matrices is denoted $\mathcal{H}^{n\times n}$, and defined as

$$\mathcal{H}^{n\times n} := \{ M \in \mathbb{C}^{n\times n} : M^* = M \}$$

**Definition:** The set of complex, normal $n \times n$ matrices is denoted $\mathcal{N}^{n\times n}$, and defined as

$$\mathcal{N}^{n\times n} := \{ M \in \mathbb{C}^{n\times n} : M^*M = MM^* \}$$

Note that

$$\mathcal{S}^{n\times n} \subset \mathcal{H}^{n\times n} \subset \mathcal{N}^{n\times n}$$
**Fact:** Hermitian matrices have real eigenvalues:

**Proof:** Let $\lambda \in \mathbb{C}$ be an eigenvalue of a Hermitian matrix $M = M^*$, and let $v \neq 0_n$ be a corresponding eigenvector, so that $Mv = \lambda v$.

Note that

$$2 \text{Re}(\lambda) \|v\|^2 = \lambda \|v\|^2 + \overline{\lambda} \|v\|^2$$

$$= v^*(\lambda v) + (\lambda v)^* v$$

$$= v^* Mv + (Mv)^* v$$

$$= v^* Mv + v^* M^* v$$

$$= v^* Mv + v^* Mv \quad \text{using } M = M^*$$

$$= 2v^* Mv$$

$$= 2\lambda \|v\|^2$$

Since $v \neq 0_n$, the norm is positive, divide out leaving

$$\text{Re}(\lambda) = \lambda$$

as desired.

**Remark:** If $M \in \mathcal{H}^{n \times n}$, the eigenvalues of $M$ are real, and can be ordered

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

and it makes sense to write

$$\lambda_{\text{max}}(M) \quad \text{and} \quad \lambda_{\text{min}}(M)$$

without confusion.
**Fact:** An upper triangular, normal matrix is actually diagonal. Check it out...

**Fact:** Given $Q \in \mathbb{C}^{n \times n}$ satisfying $Q^*Q = I_n$, then for any $M \in \mathbb{C}^{n \times n}$,

$$M \in \mathcal{N} \iff Q^*MQ \in \mathcal{N}$$

The proof is simple:

$$M^*M = MM^* \iff Q^* (M^*M) Q = Q^* (MM^*) Q \iff Q^*M^*MQ = Q^*MM^*Q \iff Q^*M^*QQ^*M^*Q = Q^*M^*QQ^*M^*Q \iff Q^*MQ \in \mathcal{N} \iff Q^*MQ \in \mathcal{N}$$

Hence,

**Fact:** A normal matrix $M$ has an orthonormal set of eigenvectors, ie., there exists a matrices $Q, \Lambda \in \mathbb{C}^{n \times n}$ with

- $Q^*Q = I_n$,
- $\Lambda$ diagonal
- $M = Q\Lambda Q^*$
If $M = M^*$, then

$$\{x^* M x : \|x\|_2 = 1\} = [\lambda_{\text{min}}(M), \lambda_{\text{max}}(M)]$$

**Proof:** Basic idea:

- Let $Q \Lambda Q^* = M$ be a Schur decomposition of $M$
- Since $M = M^*$, $\Lambda$ is diagonal and real
- Notate $\xi := Q^* x$, noting $\|Q\xi\|_2 = \|\xi\|_2$ for all $\xi$

Then

$$\{x^* M x : \|x\|_2 = 1\} = \{x^* Q \Lambda Q^* x : \|x\|_2 = 1\}$$

$$= \{\xi^* \Lambda \xi : \|Q\xi\|_2 = 1\}$$

$$= \{\xi^* \Lambda \xi : \|\xi\|_2 = 1\}$$

$$= \{\sum_{i=1}^n \lambda_i |\xi_i|^2 : \sum_{i=1}^n |\xi_i|^2 = 1\}$$

For any $\alpha \in [0, 1]$, define

$$\xi_1 := \sqrt{\alpha}, \, \xi_2 = \xi_3 = \cdots = \xi_{n+1} = 0, \, \xi_n := \sqrt{1-\alpha}$$

yielding

$$\sum_{i=1}^n \lambda_i |\xi_i|^2 = \alpha \lambda_1 + (1 - \alpha) \lambda_n$$

which shows by proper choice of $\alpha$, anything in between $\lambda_1$ and $\lambda_n$ can be achieved.

**Warning:** Take $M = M^*$. Then

$$\{x^* M x : \|x\|_2 \leq 1\} \neq [\lambda_{\text{min}}(M), \lambda_{\text{max}}(M)]$$
Now, return to expression for $\|M\|_{2→2}$.

\[
\|M\|_{2→2}^2 := \max_{\|x\|\leq 1} \|Mx\|^2
\]

\[
= \max_{\|x\|=1} \|Mx\|^2
\]

\[
= \max_{\|x\|=1} x^*M^*Mx
\]

\[
= \lambda_{\text{max}} (M^*M)
\]

Hence, $\|M\|_{2→2}$ is often denoted by $\bar{\sigma}(M)$, called the maximum singular value of $M$. Since the nonzero eigenvalues of $AB$ equal the nonzero eigenvalues of $BA$, it follows that

\[
\bar{\sigma}(M) = \bar{\sigma}(M^*)
\]
**Linear Algebra Definite Hermitian (and Symmetric) Matrices**

**Definition:** A matrix $M \in \mathcal{H}^{n \times n}$ is

1. **positive definite** (denoted $M \succ 0$) if $u^* M u > 0$ for every $u \in \mathbb{C}^n, u \neq 0_n$.

2. **positive semi-definite** (denoted $M \succeq 0$) if $u^* M u \geq 0$ for every $u \in \mathbb{C}^n$.

3. **negative definite** (denoted $M \prec 0$) if $u^* M u < 0$ for every $u \in \mathbb{C}^n, u \neq 0_n$.

4. **negative semi-definite** (denoted $M \preceq 0$) if $u^* M u \leq 0$ for every $u \in \mathbb{C}^n$.

For $A, B \in \mathcal{H}^{n \times n}$, write $A \preceq B$ if $A - B \preceq 0$. Similarly for $\prec, \succ$ and $\succeq$.

**Easy Facts:**

1. If $A \preceq B$ and $B \preceq A$, then indeed, $A = B$. If $A \preceq B$ and $C \preceq D$, then $A + C \preceq B + D$.

2. $L \in \mathbb{F}^{n \times n}$ invertible, $M \in \mathcal{H}^{n \times n}$, then

   $$M \succ 0 \iff L^* M L \succ 0$$

3. $L \in \mathbb{F}^{n \times m}$ full column rank (so $n \geq m$), $M \in \mathcal{H}^{n \times n}$, then

   $$M \succ 0 \implies L^* M L \succ 0$$

4. For any $W \in \mathbb{F}^{n \times m}$, $W^* W \succeq 0$.

5. For any $W \in \mathbb{F}^{n \times m}$, if $	ext{rank} W = m$, then $W^* W \succ 0$.

6. $M \succ 0$ if and only if $\lambda_{\min}(M) > 0$. 

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7. If \( M \in \mathcal{H}^{n \times n} \), then \( M < 0 \iff (-M) > 0 \)

8. If \( A_1, A_2 \in \mathcal{H}^{n \times n}, A_1 > 0, A_2 > 0 \), then for each \( t \in [0, 1] \),
\[
(1 - t)A_1 + tA_2 > 0
\]

9. Given \( X \in \mathcal{H}^{n \times n}, Z \in \mathcal{H}^{m \times m} \) and \( Y \in \mathbb{F}^{n \times m} \),
\[
\begin{bmatrix}
X & Y \\
Y^* & Z
\end{bmatrix} > 0 \implies X > 0, Z > 0
\]

10. \( \bar{\sigma}(\cdot) \) bounds are easily converted into definiteness relations. For any matrix \( M \in \mathbb{C}^{n \times m} \),
\[
\bar{\sigma}(M) < \beta \iff M^*M - \beta^2 I_m < 0 \\
\iff MM^* - \beta^2 I_n < 0 \\
\iff \bar{\sigma}(M^*) < \beta
\]

11. If \( M \) is invertible, and \( M^* = M \), then \( M > 0 \) if and only if \( M^{-1} > 0 \).

12. **Warning**: If \( M \neq M^* \), then \( M \) having positive, real eigenvalues does not guarantee \( x^* M x > 0 \). Instead, check \( M + M^* \), since it is Hermitian, and \( x^* M x = \frac{1}{2} x^* (M + M^*) x \). For example,
\[
M = \begin{bmatrix}
1 & 10 \\
0 & 1
\end{bmatrix}
\]

13. If \( M + M^* < 0 \), then eigenvalues of \( M \) have negative real-part

14. If \( M = M^* < 0 \), then for any \( \Delta = \Delta^* \), there is an \( \epsilon > 0 \) such that \( M + t\Delta < 0 \) for all \( |t| < \epsilon \).
**Theorem:** Let $T_{i=1}^k$ be a family of matrices, with each $T_i \in \mathbb{C}^{n \times n}$, and $T_i^* = T_i$. If there exist scalars $\{d_i\}_{i=1}^k$ with $d_i \geq 0$, and

$$T_0 - \sum_{i=1}^k d_i T_i > 0$$

then for all $x \in \mathbb{C}^n$ which satisfy $x^* T_i x > 0$ for $1 \leq i \leq k$, it follows that $x^* T_0 x > 0$.

**Proof:** Let $x \in \mathbb{C}^n$ satisfy $x^* T_i x > 0$ for all $1 \leq i \leq k$. Hence, $x \neq 0$. By hypothesis, we have

$$x^* \left[ T_0 - \sum_{i=1}^k d_i T_i \right] x > 0$$

which implies

$$x^* T_0 x > \sum_{i=1}^k d_i x^* T_i x \geq 0$$

as desired. ♦

**Remark:** Easily replace $>$ with $\geq$ in above statement.
Theorem: Given $M \in \mathbb{F}^{n \times m}$. Then there exists

- $U \in \mathbb{F}^{n \times n}$, with $U^*U = I_n$,
- $V \in \mathbb{F}^{m \times m}$, with $V^*V = I_m$,
- integer $0 \leq k \leq \min(n, m)$, and
- real numbers $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$

such that

$$M = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^*$$

where $\Sigma \in \mathbb{R}^{k \times k}$ is

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_k \end{bmatrix}$$
**Proof:** Clearly \( M^*M \in \mathcal{H}^{m \times m} \) is positive semi-definite. Since it is Hermitian, it has a full set of orthonormal eigenvectors, and the eigenvalues are real, and nonnegative. Let \( \{v_1, v_2, \ldots, v_m\} \) denote an orthonormal choice of eigenvectors, associated with the eigenvalues

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > \lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_m = 0
\]

For any \( 1 \leq j \leq m \), we have

\[
\|Mv_j\|^2 = v_j^*M^*Mv_j = \lambda_j v_j^*v_j = \lambda_j
\]

Hence, for \( j > k \), it follows that \( Mv_j = 0_n \).

For \( 1 \leq j \leq k \), define \( \sigma_j := \sqrt{\lambda_j} \). Next, for \( 1 \leq j \leq k \), define vectors \( u_j \in \mathbb{F}^n \) via

\[
u_j := \frac{1}{\sigma_j} Mv_j
\]

Note that for any \( 1 \leq j, h \leq k \),

\[
\sigma_j v_h^*M^*Mv_j = \frac{1}{\sigma_h \sigma_j} v_h^* M^* M v_j = \frac{1}{\sigma_h \sigma_j} v_h^* (\lambda_j v_j) = \frac{\sigma_j}{\sigma_h} v_h^* v_j
\]

This implies that \( u_h^*u_j = \delta_{hj} \). Hence the set \( \{u_1, \ldots, u_k\} \) are mutually orthonormal vectors in \( \mathbb{F}^n \). Using Gram-Schmidt, construct vectors \( u_{k+1}, \ldots, u_n \) to fill this out, so

\[
\{u_1, \ldots, u_k, u_{k+1}, \ldots, u_n\}
\]

is a mutually orthonormal set if vectors in \( \mathbb{F}^n \). Now we want to consider \( u_h^*Mv_j \) for 4 cases (depending on how \( h, j \) compare to \( k \).
\[ u_h^*Mv_j = \frac{1}{\sigma_h}v_h^*M^*Mv_j \]
\[ = \frac{\sigma_j}{\sigma_h}v_h^*v_l \]
\[ = \sigma_h\delta_{hj} \]

• any \( h \), with \( j > k \). Substituting gives

\[ u_h^*Mv_j = u_h^*(Mv_j) \]
\[ = u_h^*0 \]
\[ = 0 \]

• \( h > k \), and \( 1 \leq j \leq k \). Substituting gives

\[ u_h^*Mv_j = u_h^*(\sigma_ju_j) \]
\[ = \sigma_ju_h^*u_j \]
\[ = 0 \]

Defining matrices \( U \) and \( V \) with columns made up of the \( \{u_h\}_{h=1}^n \) and \( \{v_j\}_{j=1}^m \) completes the proof. \( \# \)
If \( M = M^* \succeq 0 \), then there is a unique matrix \( S \) satisfying

- \( S = S^* \)
- \( S \succeq 0 \) (moreover, \( S > 0 \iff M > 0 \))
- \( S^2 = M \)

\( S \) is called the \textit{Hermitian square-root of} \( M \) and denoted \( M^{1/2} \).

Facts:

1. Calculating the Hermitian square root of \( M \):
   
   (a) Do a Schur decomposition of \( M \), so \( M = Q\Lambda Q^* \).
   
   (b) Since \( M = M^* \), \( \Lambda \) is diagonal and real.
   
   (c) Since \( M \succeq 0 \), the diagonal entries of \( \Lambda \) are non-negative, denote them as \( \lambda_1, \lambda_2, \ldots, \lambda_n \).

   (d) Define
   
   \[
   S := Q \begin{bmatrix}
   \sqrt{\lambda_1} & 0 & \cdots & 0 \\
   0 & \sqrt{\lambda_2} & \cdots & 0 \\
   \vdots & \vdots & \ddots & \vdots \\
   0 & 0 & \cdots & \sqrt{\lambda_n}
   \end{bmatrix} Q^*
   \]

   (e) Note that \( S = S^* \succeq 0 \), and \( S^2 = M \).

2. If \( M = M^* > 0 \), then \( M \) is invertible, and \( M^{-1} \) is Hermitian and positive definite. Hence it has a Hermitian square root. In fact

\[
\left( M^{-1} \right)^{1/2} = \left( M^{1/2} \right)^{-1}
\]

so write \( M^{-1/2} \) without any confusion as to its meaning.
Fact: Given $M \in \mathcal{H}^{n \times n}$ and $L \in \mathbb{C}^{n \times n}$, with $L$ invertible. Then

$$M \succ 0 \iff L^*ML \succ 0$$

Fact: Given $X \in \mathcal{H}^{n \times n}$, $Y \in \mathcal{H}^{m \times m}$,

$$\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \succ 0 \iff X \succ 0 \text{ and } Y \succ 0$$

Fact: Given $X \in \mathcal{H}^{n \times n}$, $Z \in \mathbb{F}^{n \times m}$,

$$\begin{bmatrix} X & Z \\ Z^* & I_m \end{bmatrix} \succ 0 \iff X - ZZ^* \succ 0$$

Proof: Use $L := \begin{bmatrix} I_n & 0 \\ -Z^* & I_m \end{bmatrix}$.

This leads to what is typically called the “Schur complement” theorem.

Fact: Given $X \in \mathcal{H}^{n \times n}$, $Y \in \mathcal{H}^{m \times m}$, $Z \in \mathbb{C}^{n \times m}$,

$$\begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix} \succ 0 \iff Y \succ 0, \text{ and } X - ZY^{-1}Z^* \succ 0$$

Proof: Note that if $Y \succ 0$,

$$\begin{bmatrix} I_n & 0 \\ 0 & Y^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & Y^{-\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} X & ZY^{-\frac{1}{2}} \\ Y^{-\frac{1}{2}}Z^* & I_m \end{bmatrix}$$
Linear Algebra

More Schur Complements

Lemma: Suppose $X_{11} \in \mathbb{F}^{n \times n}$, $Y_{11} \in \mathbb{F}^{n \times n}$, with $X_{11} = X_{11}^* \succ 0$, and $Y_{11} = Y_{11}^* \succ 0$. Let $r$ be a non-negative integer. Then there exist $X_{12} \in \mathbb{F}^{n \times r}$, $X_{22} \in \mathbb{F}^{r \times r}$ such that $X_{22} = X_{22}^*$, and

$$
\begin{bmatrix}
X_{11} & X_{12} \\
X_{12}^* & X_{22}
\end{bmatrix} \succ 0 \quad \text{and} \quad 
\begin{bmatrix}
X_{11} & X_{12} \\
X_{12}^* & X_{22}
\end{bmatrix}^{-1} = 
\begin{bmatrix}
Y_{11} & ? \\
? & ?
\end{bmatrix}
$$

if and only if

$$
\begin{bmatrix}
X_{11} & I_n \\
I_n & Y_{11}
\end{bmatrix} \succeq 0 \quad \text{and} \quad \text{rank} \begin{bmatrix}
X_{11} & I_n \\
I_n & Y_{11}
\end{bmatrix} \leq n + r
$$

These last two conditions are equivalent to $X_{11} \succeq Y_{11}^{-1}$ and $\text{rank} (X_{11} - Y_{11}^{-1}) \leq r$.

Proof: Apply Schur Complement and Matrix inversion Lemmas...

$\Leftarrow$ By assumption, there is a matrix $L \in \mathbb{F}^{n \times r}$ such that $X_{11} - Y_{11}^{-1} = LL^*$. Defining $X_{12} := L$, and $X_{22} := I_r$ and note that

$$
\begin{bmatrix}
X_{11} & L \\
L^* & I_r
\end{bmatrix}^{-1} = 
\begin{bmatrix}
(X_{11} - LL^*)^{-1} & -(X_{11} - LL^*)^{-1} L \\
-L^* (X_{11} - LL^*)^{-1} & L^*(X_{11} - LL^*)^{-1} L + I_r
\end{bmatrix} = 
\begin{bmatrix}
Y_{11} & ? \\
? & ?
\end{bmatrix}
$$

$\Rightarrow$ Using the matrix inversion lemma (item 1), it must be that

$$
Y_{11}^{-1} = X_{11} - X_{12}X_{22}^{-1}X_{12}^*.
$$

Hence, $X_{11} - Y_{11}^{-1} = X_{12}X_{22}^{-1}X_{12}^* \succeq 0$, and indeed,

$$
\text{rank} (X_{11} - Y_{11}^{-1}) = \text{rank} (X_{12}X_{22}^{-1}X_{12}^*) \leq r.
$$

The other rank condition follows because

$$
\begin{bmatrix}
I_n & -Y_{11}^{-1} \\
0 & I_n
\end{bmatrix} 
\begin{bmatrix}
X_{11} & I_n \\
I_n & Y_{11}
\end{bmatrix} 
\begin{bmatrix}
I_n & 0 \\
-Y_{11}^{-1} & I_n
\end{bmatrix} = 
\begin{bmatrix}
X_{11} - Y_{11}^{-1} & 0 \\
0 & Y_{11}
\end{bmatrix}
$$
Lots of the control design algorithms we will study ($\mathcal{H}_\infty$, for instance) hinge on the following result from linear algebra:

1. Given $R \in \mathbb{F}^{l \times l}$, $U \in \mathbb{F}^{l \times m}$ and $V \in \mathbb{F}^{p \times l}$, where $m, p \leq l$.
2. We want to minimize $\bar{\sigma} [R + UQV]$ over $Q \in \mathbb{F}^{m \times p}$.

3. Suppose $U_\perp \in \mathbb{F}^{l \times (l-m)}$ and $V_\perp \in \mathbb{F}^{(l-p)\times l}$ have
   - $[U \ U_\perp], \begin{bmatrix} V \\ V_\perp \end{bmatrix}$ are both invertible
   - $U^*U_\perp = 0_{m \times (l-m)}$, $VV^*_\perp = 0_{p \times (l-p)}$

Then

$$\inf_{Q \in \mathbb{F}^{m \times p}} \bar{\sigma} [R + UQV] < 1$$

if and only if

- $V_\perp (R^*R - I) V^*_\perp \prec 0$
- $U^*_\perp (RR^* - I) U_\perp \prec 0$

**Remark:** Essentially, $R$ must be smaller than 1 on the directions that $U$ and $V$ are perpendicular to.
Matrix dilation problems are of the form:

*Given a partially specified matrix - when can the unspecified elements be chosen so that the full matrix has some property?*

Already seen one type of problem. Next, we derive a main elementary matrix dilation theorem. We start simple and build...

Given \( A \in \mathbb{C}^{m \times n} \), it is clear that

\[
\min_{X \in \mathbb{C}^{q \times n}} \bar{\sigma} \begin{bmatrix} X \\ A \end{bmatrix} = \bar{\sigma}(A)
\]

and this can easily be achieved by choosing \( X := 0 \). Pick some \( \gamma > \bar{\sigma}(A) \). Characterize all \( X \) that give \( \bar{\sigma} \begin{bmatrix} X \\ A \end{bmatrix} < \gamma \).

**Lemma:** Suppose \( Y \in \mathbb{F}^{n \times n} \) is invertible. Then

\[
\{ X \in \mathbb{F}^{q \times n} : X^*X < Y^*Y \} = \{ WY : W \in \mathbb{F}^{q \times n}, \bar{\sigma}(W) < 1 \}
\]

**Proof:**

A simple chain of equivalences

\[
X^*X < Y^*Y \iff X^*X - Y^*Y < 0 \\
\iff Y^{-*} [X^*X - Y^*Y] Y^{-1} < 0 \\
\iff Y^{-*} X^*X Y^{-1} - I < 0 \\
\iff \bar{\sigma}(XY^{-1}) < 1 \\
\iff \bar{\sigma}(W) < 1 \text{ and } W = XY^{-1} \\
\iff \bar{\sigma}(W) < 1 \text{ and } X = WY
\]
The lemma easily gives

**Lemma:** Given $A \in \mathbb{F}^{m \times n}$, and $\gamma > \bar{\sigma}(A)$. Then

$$
\left\{ X \in \mathbb{F}^{q \times n} : \bar{\sigma} \left[ \begin{array}{c} X \\ A \end{array} \right] < \gamma \right\} = \left\{ W \left( \gamma^2 I_n - A^* A \right)^{\frac{1}{2}} : W \in \mathbb{F}^{q \times n}, \bar{\sigma}(W) < 1 \right\}
$$

**Proof:**

Another chain of equivalences

$$
\bar{\sigma} \left[ \begin{array}{c} X \\ A \end{array} \right] < \gamma \iff X^*X + A^*A - \gamma^2 I < 0 \\
\iff X^*X < \gamma^2 I - A^*A \\
\iff X^*X < (\gamma^2 I - A^*A)^{1/2} (\gamma^2 I - A^*A)^{1/2}
$$

Now apply previous Lemma.

Equivalently, for any $X \in \mathbb{F}^{q \times n}$ and $\gamma > \bar{\sigma}(A)$, we have

$$
\bar{\sigma} \left[ \begin{array}{c} X \\ A \end{array} \right] < \gamma \iff \bar{\sigma} \left[ X \left( \gamma^2 I_n - A^* A \right)^{-\frac{1}{2}} \right] < 1
$$
Similarly, for $B \in \mathbb{F}^{q \times p}$, and $\gamma > \bar{\sigma}(B)$, we have

$$\left\{ X \in \mathbb{F}^{q \times n} : \bar{\sigma} \left[ \begin{array}{c} X \\ B \end{array} \right] < \gamma \right\} =$$

$$\left\{ (\gamma^2 I_q - BB^*)^{\frac{1}{2}} W : W \in \mathbb{F}^{q \times n}, \bar{\sigma}(W) < 1 \right\}$$

Along these lines, a corollary follows:

**Corollary RV:** Given $R \in \mathbb{F}^{n \times n}$, $V \in \mathbb{F}^{t \times n}$, with $V$ full row rank. Then

$$\min_{Q \in \mathbb{F}^{n \times t}} \bar{\sigma}(R + QV) = \bar{\sigma}(RV^\perp)$$

where $V^\perp \in \mathbb{F}^{(n-t) \times n}$ satisfies

$$V^\perp V^* = I_{n-t}, \quad V^\perp V^* = 0, \quad \det \begin{bmatrix} V \\ V^\perp \end{bmatrix} \neq 0$$
**Proof:** let $S \in \mathbb{F}^{t \times t}$ be invertible such that $V_o := SV \in \mathbb{F}^{t \times n}$ satisfies $V_oV_o^* = I_t$. Then, for any $Q \in \mathbb{F}^{n \times t}$, we have

$$R + QV = R + QS^{-1}SV = R + QS^{-1}V_o$$

Since $S$ is invertible, by picking $Q$, we equivalently have complete freedom in picking $Q_o(:= QS^{-1})$. Hence

$$\min_{Q \in \mathbb{F}^{n \times t}} \bar{\sigma}(R + QV) = \min_{Q_o \in \mathbb{F}^{n \times t}} \bar{\sigma}(R + Q_oV_o) =$$

Also,

$$T := \begin{bmatrix} V_o \\ V_\perp \end{bmatrix}$$

is a square, unitary matrix. Hence,

$$\min_{Q_o \in \mathbb{F}^{n \times t}} \bar{\sigma}(R + Q_oV_o) = \min_{Q_o \in \mathbb{F}^{n \times t}} \bar{\sigma}((R + Q_oV_o)T^*)$$

But $(R + Q_oV_o)T^*$ is simply

$$(R + Q_oV_o)T^* = \begin{bmatrix} RV_o^* + Q_o & RV_\perp^* \end{bmatrix}$$

The minimum (over $Q_o$) that the maximum singular value can take on is clearly $\bar{\sigma}(RV_\perp^*)$, which is achieved when

$$Q_o := -RV_o^* = -RV^*S^*$$

and hence

$$Q = Q_oS = -RV^*S^*S = -RV^*(VV^*)^{-1}$$
Linear Algebra Dilatation Main Result

Given $A \in \mathbb{F}^{m \times n}$, $B \in \mathbb{F}^{q \times p}$, $C \in \mathbb{F}^{m \times p}$, what is

$$\min_{X \in \mathbb{F}^{q \times n}} \bar{\sigma} \begin{bmatrix} X & B \\ A & C \end{bmatrix}$$

The theorem, independently (and in many different forms) by Sarason, Adamjan-Arov-Krien, Sz Nagy-Foias, Davis-Kahan-Weinberger, and Parrot is:

**Theorem:** Given $A$, $B$ and $C$ as above. Then

$$\min_{X \in \mathbb{F}^{q \times n}} \bar{\sigma} \begin{bmatrix} X & B \\ A & C \end{bmatrix} = \max \left\{ \bar{\sigma} \begin{bmatrix} A & C \end{bmatrix}, \bar{\sigma} \begin{bmatrix} B \end{bmatrix} \right\}$$

**Remark:** $X = 0$ typically does not achieve the minimum cost. Try a simple, real $2 \times 2$ example...

Note that the $2 \times 2$ block matrix can be written as

$$\begin{bmatrix} X & B \\ A & C \end{bmatrix} = \begin{bmatrix} 0 & B \\ A & C \end{bmatrix} + \begin{bmatrix} I_q \\ 0 \end{bmatrix} X \begin{bmatrix} I_n & 0 \end{bmatrix}$$

which is a special form of the $R + UQV$ expression.
**Theorem:** Given $A \in \mathbf{F}^{m \times n}$, $B \in \mathbf{F}^{q \times p}$, $C \in \mathbf{F}^{m \times p}$. Then

$$\min_{X \in \mathbf{F}^{q \times n}} \tilde{\sigma} \begin{bmatrix} X & B \\ A & C \end{bmatrix} = \max \left\{ \tilde{\sigma} \begin{bmatrix} A & C \end{bmatrix}, \tilde{\sigma} \begin{bmatrix} B \end{bmatrix} \right\}$$

**Proof:** Clearly, nothing smaller than the right-hand-side is achievable. Take any $\gamma > \tilde{\sigma} \begin{bmatrix} A & C \end{bmatrix}$. Then

$$\min_{X} \tilde{\sigma} \begin{bmatrix} X & B \\ A & C \end{bmatrix} < \gamma \iff \min_{X} \tilde{\sigma} \left( \begin{bmatrix} X & B \end{bmatrix} S^{-\frac{1}{2}} \right) < 1$$

where

$$S := \gamma^2 I - \begin{bmatrix} A^* \\ C^* \end{bmatrix} \begin{bmatrix} A & C \end{bmatrix}$$

Hence there exists an $X$ such that $\tilde{\sigma} \begin{bmatrix} X & B \\ A & C \end{bmatrix} < \gamma$ if and only if

$$\min_{X} \tilde{\sigma} \begin{bmatrix} X \\ Q \end{bmatrix} \begin{bmatrix} I & 0 \\ V \end{bmatrix} S^{-\frac{1}{2}} + \begin{bmatrix} 0 & B \end{bmatrix} \begin{bmatrix} R \end{bmatrix} S^{-\frac{1}{2}} < 1$$

What should $V_\perp$ be? It needs to satisfy $V_\perp V^* = 0$ and $V_\perp V^*_\perp = I$. The first condition implies that

$$V_\perp V^* = 0 \iff V_\perp S^{-\frac{1}{2}} \begin{bmatrix} I \\ 0 \end{bmatrix} = 0$$

so that $V_\perp$ is of the form $V_\perp = \begin{bmatrix} 0 & L \end{bmatrix} S^{\frac{1}{2}}$ for some (at this point) arbitrary $L$. The second condition requires

$$V_\perp V^* = I \iff L \left( \gamma^2 I - C^* C \right) L^* = I$$

so that $L = (\gamma^2 I - C^* C)^{-\frac{1}{2}}$ is a suitable choice.
Hence, the original equivalence continues,

\[
\min_X \bar{\sigma} (QV + R) < 1 \iff \bar{\sigma} (RV_{\perp}) < 1 \\
\iff \bar{\sigma} \left[ B \left( \gamma^2 I - C^*C \right)^{-\frac{1}{2}} \right] < 1 \\
\iff \bar{\sigma} \left[ \begin{bmatrix} B \\ C \end{bmatrix} \right] < \gamma
\]

Hence, any \( \gamma \) larger than both \( \bar{\sigma} [A \ C] \) and \( \bar{\sigma} \left[ \begin{bmatrix} B \\ C \end{bmatrix} \right] \) is achievable, using, for instance

\[
X := -B \left( \gamma^2 I - C^*C \right)^{-1} C^*A
\]

Moreover (though we do not explicitly use it) the minimum is achieved (compactness argument).
Partial answer to the $R + UQV$ problem when similarity scalings are included:

1. Let $R, U, V, U_\perp$ and $V_\perp$ be given as before.

2. Let $\mathcal{Z} \subset \mathbb{F}^{l \times l}$ be a given set of positive definite, Hermitian matrices

Then

$$\inf_{\substack{Q \in \mathbb{F}^{m \times p} \\bar{\sigma} \left( Z^{1/2} (R + UQV) Z^{-1/2} \right) < 1 \atop Z \in \mathcal{Z}}}$$

if and only if there is a $Z \in \mathcal{Z}$ such that

$$V_\perp (R^* Z R - Z) V_\perp^* < 0$$

and

$$U_\perp^* (R Z^{-1} R^* - Z^{-1}) U_\perp < 0.$$
Proof: For each fixed \( Z \in \mathcal{Z} \), consider the problem

\[
\beta(Z) := \inf_{Q \in \mathbb{F}_{r \times t}} \sigma \left[ Z^{\frac{1}{2}} (R + UQV) Z^{-\frac{1}{2}} \right]
\]

Define \( \tilde{R} := Z^{\frac{1}{2}} R Z^{-\frac{1}{2}} \), \( \tilde{U} := Z^{\frac{1}{2}} U \), \( \tilde{V} = V Z^{-\frac{1}{2}} \). Note that the columns of \( Z^{-\frac{1}{2}} U \) span the space orthogonal to the range (column) of \( \tilde{U} \), since \( (Z^{-\frac{1}{2}} U) \) \( \tilde{U} = 0 \). Similarly, the rows of \( V Z^{-\frac{1}{2}} \) span the space orthogonal to the range (row) of \( \tilde{V} \). Therefore, for fixed \( Z \in \mathcal{Z} \), \( \beta(Z) < \alpha \) if and only if

\[
U_{\perp}^* Z^{-\frac{1}{2}} \left( Z^{\frac{1}{2}} R Z^{-\frac{1}{2}} Z^{-\frac{1}{2}} R^* Z^{\frac{1}{2}} \right) Z^{-\frac{1}{2}} U_{\perp} \prec 0,
\]

and

\[
V_{\perp} \left( Z^{-\frac{1}{2}} R^* Z \right) V_{\perp}^* \prec 0.
\]

These simplify to

\[
U_{\perp}^* \left( R Z^{-1} R^* - \alpha^2 Z^{-1} \right) U_{\perp} \prec 0, \tag{1}
\]

and

\[
V_{\perp} \left( R^* Z R - \alpha^2 Z \right) V_{\perp}^* \prec 0 \tag{2}
\]

as claimed. \#
The previous results are directly useful in discrete-time problems. Using similar techniques, the analogous theorem for definiteness can be proven:

**Theorem:** Given \( R \in \mathbb{F}^{l \times l}, U \in \mathbb{F}^{l \times m} \) and \( V \in \mathbb{F}^{p \times l} \), where \( m, p \leq l \). Suppose \( U_\perp \in \mathbb{F}^{l \times (l-m)} \) and \( V_\perp \in \mathbb{F}^{(l-p) \times l} \) have

- \([U \ U_\perp], [V \ V_\perp] \) are both invertible
- \( U^*U_\perp = 0_{m \times (l-m)}, VV_\perp^* = 0_{p \times (l-p)} \)

Then, there exist a \( Q \in \mathbb{F}^{m \times p} \) such that

\[
[R + UQV] + [R + UQV]^* < 0
\]

if and only if

\[
U_\perp^* (R + R^*) U_\perp < 0, \quad V_\perp (R + R^*) V_\perp^* < 0
\]
**Completion of squares**

**Lemma:** \( S = S^* \succ 0, \ T \) given square matrices. For every \( K \),
\[
-TK^* - KT^* + KSK \succeq -TS^{-1}T^*.
\]
Furthermore, \( K_0 := TS^{-1} \) achieves equality.

**Proof:** Complete squares as
\[
-TK^* - KT^* + KSK = \left( KS^{1/2} - TS^{-1/2} \right) \left( KS^{1/2} - TS^{-1/2} \right)^* - TS^{-1}T^* \\
\succeq -TS^{-1}T^*
\]
Note that equality is achieved by making \( KS^{1/2} - TS^{-1/2} = 0 \), which can be accomplished with \( K = TS^{-1} \).

**Lemma:** \( S = S^* \succeq 0, \text{Ker} S \subseteq \text{Ker} T \). Let \( K_0 \) be any solution of the equation \( K_0 S = T \). Then for every \( K \)
\[
-TK^* - KT^* + KSK \succeq -TK_0^* - K_0 T^* + K_0 S K_0 \left( = -K_0 S K_0 \right)
\]

**Proof:** For any \( K \),
\[
T (K_0 - K)^* + (K_0 - K) T^* - K_0 S K_0^* + KSK \\
= (K_0 - K) S (K_0 - K)^* \\
\succeq 0
\]
To verify the equality, simply substitute for \( T \). Also note that the equation \( K_0 S = T \) may have many solutions. If \( K_{0,1} \) and \( K_{0,2} \) are two such solutions, then by making the argument twice above, we have
\[
K_{0,1} S K_{0,1}^* = K_{0,2} S K_{0,2}^*
\]
Equivalently, \( TK_{0,1} = TK_{0,2} \).