6 General Performance Objective

Most problems will have many exogenous inputs and many errors, for example

\[ \begin{bmatrix} \text{tracking error} \\ \text{control input} \end{bmatrix} = T \begin{bmatrix} \text{reference} \\ \text{external force} \\ \text{noise} \end{bmatrix} \]

Definition: Good Performance ⇔ gain of \( T \) is “small”

Since closed-loop system is a MIMO dynamical system, there are two aspects to “gain”

- Spatial (vector disturbances and vector errors)
- Temporal (dynamical relationship between input/output signals)

In any performance criterion, we must account for

- Relative magnitude of outside influences
Relative importance of the magnitudes of regulated variables

Performance objective must be a weighted norm

\[ \| W_L T W_R \| \]

\( W_L \) and \( W_R \) can be frequency dependent, to account for bandwidth constraints and spectral content of exogenous signals. Within the \( \mu \) setting, the most natural (mathematically) manner to characterize acceptable performance is with the MIMO \( \mathcal{H}_\infty \) norm.

### 6.1 Using Norms to Characterize Performance

Suppose \( T \) is a MIMO stable linear system, with transfer function matrix \( T(s) \). For a given driving signal \( \tilde{d}(t) \), define \( \tilde{e} \) as the output, as shown below

\[
\begin{array}{c}
\tilde{e} \\
T \\
\tilde{d}
\end{array}
\]

Assume the dimensions of \( T \) are \( n_e \times n_d \). Let \( \beta > 0 \) be defined as

\[
\beta := \| T \|_\infty := \max_{\omega \in \mathbb{R}} \bar{\sigma} [ T(j\omega) ]
\]

Now consider a response, starting from initial condition equal to 0. Parseval’s theorem gives that

\[
\frac{\| \tilde{e} \|_2}{\| \tilde{d} \|_2} := \frac{\left[ \int_0^\infty \tilde{e}^T(t)\tilde{e}(t)dt \right]^{1/2}}{\left[ \int_0^\infty \tilde{d}^T(t)\tilde{d}(t)dt \right]^{1/2}} \leq \beta
\]

Moreover, there are specific disturbances \( d \) that result in

\[
\frac{\| \tilde{e} \|_2}{\| \tilde{d} \|_2} \approx \beta
\]

Hence, \( \| T \|_\infty \) is referred to as “the \( \mathcal{L}_2 \) (or RMS) gain of the system.”
6.2 $\mathcal{H}_\infty$ Norm: Sinusoidal Interpretation

A sinusoidal, steady-state interpretation of $\|T\|_\infty$ is also possible:

For any frequency $\bar{\omega} \in \mathbb{R}$, any vector of amplitudes $a \in \mathbb{R}^{n_d}$, and any vector of phases $\phi \in \mathbb{R}^{n_d}$, with $\|a\|_2 \leq 1$, define a time signal

$$d(t) = \begin{bmatrix} a_1 \sin (\bar{\omega}t + \phi_1) \\ \vdots \\ a_{n_d} \sin (\bar{\omega}t + \phi_{n_d}) \end{bmatrix}$$

Apply this input to the system $T$, resulting in a steady-state response $\tilde{e}_{ss}$ of the form

$$\tilde{e}_{ss}(t) = \begin{bmatrix} b_1 \sin (\bar{\omega}t + \psi_1) \\ \vdots \\ b_{n_e} \sin (\bar{\omega}t + \psi_{n_e}) \end{bmatrix}$$

The vector $b \in \mathbb{R}^{n_e}$ will satisfy $\|b\|_2 \leq \beta := \|T\|_\infty$.

Moreover, $\beta$ is the smallest number such this fact is true for every $\|a\|_2 \leq 1$, $\bar{\omega}$, and $\phi$.

In this interpretation:

- Vectors of the sinusoidal magnitude responses are unweighted
- Normalized to be of unit Euclidean norm.

If realistic multivariable performance objectives are to be represented by a single, MIMO $\|\cdot\|_\infty$ objective on a closed-loop transfer function, additional scalings are necessary.

- Many different objectives are being lumped into one matrix and the “cost function” is the norm of the matrix
• Use frequency-dependent weighting functions, so that different requirements can be meaningfully combined into a single cost function.

• Diagonal weights are most easily interpreted.

Consider

\[
\begin{array}{c}
e \\
W_L \quad \tilde{e} \\
T \\
W_R \quad \tilde{d} \\
d
\end{array}
\]

Assume \( W_L \) and \( W_R \) are diagonal, stable transfer function matrices, with diagonal entries denoted \( L_i \) and \( R_i \).

\[
W_L = \begin{bmatrix} L_1 & 0 & 0 \\ 0 & \ddots & : \\ 0 & \cdots & L_{n_e} \end{bmatrix}, \quad W_R = \begin{bmatrix} R_1 & 0 & 0 \\ 0 & \ddots & : \\ 0 & \cdots & R_{n_d} \end{bmatrix}
\]

### 6.3 Weighted Sinusoids

Bounds on \( \| W_L T W_R \|_\infty \) imply bounds about the sinusoidal steady-state behavior of the signals \( \tilde{d} \) and \( \tilde{e}(= T\tilde{d}) \). Specifically, for sinusoidal signal \( \tilde{d} \), the steady-state relationship between \( \tilde{e}(= T\tilde{d}) \), \( \tilde{d} \) and \( \| W_L T W_R \|_\infty \) is as follows:

The steady-state solution \( \tilde{e}_{ss} \), denoted as

\[
\tilde{e}_{ss}(t) = \begin{bmatrix} \tilde{e}_1 \sin(\bar{\omega}t + \psi_1) \\ \vdots \\ \tilde{e}_{n_e} \sin(\bar{\omega}t + \psi_{n_d}) \end{bmatrix}
\]

satisfies \( \sum_{i=1}^{n_e} |L_i(j\bar{\omega})\tilde{e}_i|^2 \leq 1 \) for all sinusoidal input signals \( \tilde{d} \) of the form

\[
\tilde{d}(t) = \begin{bmatrix} \tilde{d}_1 \sin(\bar{\omega}t + \phi_1) \\ \vdots \\ \tilde{d}_{n_d} \sin(\bar{\omega}t + \phi_{n_d}) \end{bmatrix}
\]
satisfying
\[ \sum_{i=1}^{n_d} \frac{|\tilde{d}_i|^2}{|R_i(j\bar{\omega})|^2} \leq 1 \]

if and only if \( \|W_L W_R\|_\infty \leq 1 \).

### 6.4 Incorrect (but useful) Interpretation, Weighted Sinusoids

Hence, we approximately (not correct!!!) can say
\[ \|W_L W_R\|_\infty \leq 1 \]
if and only if

for every fixed frequency \( \bar{\omega} \), and all sinusoidal disturbances \( \tilde{d} \) satisfying
\[ |\tilde{d}_i| \leq |R_i(j\bar{\omega})| \]

the steady-state error components will satisfy
\[ |\tilde{e}_i| \leq \frac{1}{|L_i(j\bar{\omega})|} \]

Pick performance weights to reflect the desired frequency-dependent performance objective

- use \( R \) to represent the relative magnitude of sinusoids disturbances that might be present, and
- use \( \frac{1}{L} \) to represent the desired upper bound on the subsequent errors that are produced.

*Remember though,* the weighted \( \mathcal{H}_\infty \) norm \( \|W_L W_R\|_\infty < 1 \) does not actually give element-by-element bounds on the components of \( \tilde{e} \) based on element-by-element bounds on the components of \( \tilde{d} \). The precise bound is in terms of Euclidean norms of the components of \( \tilde{e} \) and \( \tilde{d} \) (as described earlier)
6.5 General Performance Specifications

Closed-Loop performance objectives are weighted closed-loop transfer functions which are to be made small through feedback.

Here is an example which includes many relevant terms

\[
\begin{align*}
\dot{d_1} & \rightarrow W_{cmd} & d_1 \\
\end{align*}
\]

\[
\begin{align*}
W_{act} & \rightarrow e_1 \\
W_{model} & \rightarrow d_2 \\
\end{align*}
\]

\[
\begin{align*}
W_{dist} & \rightarrow \tilde{d}_2 \\
G & \rightarrow \tilde{e}_1 \\
H_{sens} & \rightarrow \tilde{e}_2 \\
\Sigma & \rightarrow \tilde{e}_3 \\
W_{snois} & \rightarrow d_3 \\
\end{align*}
\]

Mathematical objective of $\mathcal{H}_\infty$ control is to make the closed-loop MIMO transfer function $T_{ed}$ satisfy

$$\|T_{ed}\|_\infty < 1$$

Weighting functions are used to scale the input/output transfer functions such that when $\|T_{ed}\|_\infty < 1$, the relationship between $\tilde{d}$ and $\tilde{e}$ is “suitable”

The interpretation of the signals are
### Table 6.6 Interpretation of Weights and Models

<table>
<thead>
<tr>
<th>Signal</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1$</td>
<td>Normalized Reference Command</td>
</tr>
<tr>
<td>$\tilde{d}_1$</td>
<td>Typical Reference Command</td>
</tr>
<tr>
<td>$d_2$</td>
<td>Normalized Exogenous Disturbances</td>
</tr>
<tr>
<td>$\tilde{d}_2$</td>
<td>Typical Exogenous Disturbances</td>
</tr>
<tr>
<td>$d_3$</td>
<td>Normalized Sensor Noise</td>
</tr>
<tr>
<td>$\tilde{d}_3$</td>
<td>Typical Sensor Noise</td>
</tr>
<tr>
<td>$e_1$</td>
<td>Weighted Control Signals</td>
</tr>
<tr>
<td>$\tilde{e}_1$</td>
<td>Actual Control Signals</td>
</tr>
<tr>
<td>$e_2$</td>
<td>Weighted Tracking Errors</td>
</tr>
<tr>
<td>$\tilde{e}_2$</td>
<td>Actual Tracking Errors</td>
</tr>
<tr>
<td>$e_3$</td>
<td>Weighted Plant Errors</td>
</tr>
<tr>
<td>$\tilde{e}_3$</td>
<td>Actual Plant Errors</td>
</tr>
</tbody>
</table>

**$W_{cmd}$**

- Used in problems requiring tracking of a reference command.
- $W_{cmd}$ shapes (magnitude and frequency) the normalized reference command signals into the actual (or typical) reference signals that we expect to occur.
- In typical servo-problems, $W_{cmd}$ is flat at low frequency and rolls off at high frequency.
- For example, in a flight control problem, fighter pilots can (and will) generate stick input reference commands up to a bandwidth of about 2Hz. Say the stick has maximum travel of 3 inches. Pilot commands would then be modeled as normalized signals passed through a first order filter

\[
W_{cmd} = \frac{3}{\frac{1}{2.2\pi}s + 1}
\]
$W_{model}$

- Represents a desired ideal model for the closed-loop system
- Used in problems with tracking requirements.
- Example: for “good” command tracking response, we might desire our closed-loop system to respond as well damped second-order system, so choose specific $\omega$ and $\zeta$ and define

\[
W_{model} = \frac{\omega^2}{s^2 + 2\zeta\omega + \omega^2}
\]

- Example: Unit conversions might be necessary too. In the fighter pilot example, suppose roll-rate is being commanded, and 10°/second response is desired for each inch of stick motion. In these units, appropriate model is

\[
W_{model} = 10\frac{\omega^2}{s^2 + 2\zeta\omega + \omega^2}
\]

$W_{dist}$

- Shapes the frequency content and magnitude of the exogenous disturbances effecting the plant
- Example: airplane – dominant performance objective: maintain pilot’s desired heading in the face of gust disturbances.
- Example: precision instrument – dominant performance objective: mechanically isolate the microscope from outside mechanical disturbances, such as the ground excitations, pressure waves, and air currents
- The spectrum and relative magnitudes of these disturbances are captured in the transfer function weighting matrix $W_{dist}$.

$W_{perf1}$ and $W_{perf2}$. $W_{perf1}$ weights the difference between the response of the plant and the response of the ideal model, $W_{model}$. Often we desire

- accurate matching of the ideal model at low frequency
while requiring less accurate matching at higher frequency

The inverse of the weight should be related to the allowable size of tracking errors, in the face of the reference commands and disturbances described by $W_{ref}$ and $W_{dist}$.

$W_{perf}$ penalizes variables internal to the process $G$, such as

- actuator states which are internal to $G$, or
- other variables that are not part of the tracking objective.

$W_{act}$

- Used to shape the penalty on control signal usage
- Penalize limits the deflection/position, deflection rate/velocity, etc., response of the control signals, in the face of the tracking and disturbance rejection objectives already defined
- Each control signal is usually penalized independently.

$W_{snois}$

- Represents frequency content of sensor noise
- Derived from laboratory experiments or based on manufacturer measurements
- Example: medium grade accelerometers have substantial noise at low frequency and high frequency. Therefore the corresponding $W_{snois}$ weight would be larger at low and high frequency and have a smaller magnitude in the mid-frequency range.
- Example: Displacement or rotation measurements are often quite accurate at low frequency or in steady-state but respond poorly as frequency increases. Weighting function for this sensor would be small at low frequency, gradually increase in magnitude as a first or second system and level out at high frequency.
$H_{sens}$

- Not a weight - actually part of the feedback loop.
- Represents a model of the sensor dynamics or an external anti-aliasing filter
- Based on physical characteristics of the individual sensor components