4 Robust Performance of SISO Feedback Systems

In this section, we investigate a simple robust performance problem. By itself, the problem we consider is limited, and may not apply in many situations. A significant part of the course will be devoted to generalizing the results here to more general situations.

4.1 Uncertainty Modeling

We are given a nominal plant, \( P \in \mathcal{R} \); a controller \( C \in \mathcal{R} \) with \( C \) stabilizing \( P \); an uncertainty weighting function \( W_u \in \mathcal{S} \); and a performance weighting function \( W_p \in \mathcal{S} \).

In order to characterize and mathematically describe uncertainty in the plant model \( P \), we define a “set” of plants centered at \( P \) by

\[
M (P, W_u) := \{ P (1 + W_u \Delta) \in \mathcal{R} : \Delta \in \mathcal{R}, \quad \sup_{\omega \in \mathbb{R}} |\Delta(j\omega)| < 1, \#rhp (P) = \#rhp (P (1 + W_u \Delta)) \}
\]

This is called a multiplicative uncertainty model. It clearly depends both on \( P \) and \( W_u \). In this formulation, the quantities \( P, W_u \) and \( \Delta \) do not represent 3 separate physical devices (ie., components), but rather the formula \( P(1 + W_u \Delta) \), after exact cancellations, represents the single plant component.

For example, consider \( P = \frac{1}{s - 1} \), \( W_u = \frac{1}{2} \) and three possible plants

\[
\tilde{P}_1 := \frac{1}{s - 1.1}, \quad \tilde{P}_2 := \frac{s + 4}{(s - 0.9)(s + 4.1)}, \quad \tilde{P}_3 := \frac{s - 5}{(s - 1)(s - 5.1)}
\]

By choosing \( \Delta_i \in \mathcal{R} \) as

\[
\Delta_1 := \frac{0.2}{s - 1.1}, \quad \Delta_2 := \frac{-0.4s - 0.61}{(s + 4.1)(s - 0.9)}, \quad \Delta_3 := \frac{0.2}{s - 5.1}
\]

it is easy to check that in each case

\[
\sup_{\omega \in \mathbb{R}} |\Delta(j\omega)| < 1, \quad \text{and} \quad \tilde{P}_i = P (1 + W_u \Delta_i)
\]
Does that mean each $\tilde{P}_i \in M(P, W_u)$? The answer is no. We must furthermore check that the number of right-half plane poles of $\tilde{P}_i$ is equal to the number of right-half plane poles of $P$. Clearly, $\tilde{P}_1$ and $\tilde{P}_2$ satisfy this, though $\tilde{P}_3$ does not. Hence

$$\tilde{P}_1, \tilde{P}_2 \in M(P, W_u), \quad \tilde{P}_3 \notin M(P, W_u)$$

The next lemma shows that it is correct to think of $|W_u(j\omega)|$ as the “percentage uncertainty” in $P(j\omega)$. This is a useful interpretation of $W_u$.

**Lemma 38** Take $\bar{\omega} \in \mathbb{R}$, and suppose that $j\bar{\omega}$ is not a pole or zero of $P$. Let $\gamma \in \mathbb{C}$, with $|\gamma| < |P(j\bar{\omega})W(j\bar{\omega})|$. Then, there is a $\tilde{P} \in M(P, W_u)$ such that $\tilde{P}(j\bar{\omega}) = P(j\bar{\omega}) + \gamma$. Conversely, for any $\tilde{P} \in M(P, W_u)$ we have $|\tilde{P}(j\bar{\omega}) - P(j\bar{\omega})| < |P(j\bar{\omega})W_u(j\bar{\omega})|$.

**Remark:** So, at each frequency $\omega$, there is a disk, centered at $P(j\omega)$, with radius $|P(j\omega)W_u(j\omega)|$ that represents the possible values of $\tilde{P}(j\omega)$.

**Proof:** Since $0 \leq |\gamma| < |P(j\bar{\omega})W_u(j\bar{\omega})|$, it follows that

$$\delta := \frac{\gamma}{P(j\bar{\omega})W_u(j\bar{\omega})}$$

is well defined, and $|\delta| < 1$. Use the results from problem 4 in this section, to create a $\Delta \in \mathcal{S}$ such that

$$\|\Delta\|_\infty = |\delta| < 1, \quad \Delta(j\bar{\omega}) = \delta$$

Define $\tilde{P} := P(1 + W_u \Delta)$, note that $\tilde{P}(j\bar{\omega}) = P(j\bar{\omega}) + \gamma$. The last thing to check is that the number of right-half plane poles of $\tilde{P}$ is the same as $P$. Since $W_u$ and $\Delta$ are both in $\mathcal{S}$, $\tilde{P}$ certainly does not have more right-half plane poles than $P$. However, the choice of $\Delta$ may cause zeros of $1 + W_u \Delta$ to cancel some right-half plane poles of $P$. But, $\Delta$ can be modified so that this does not happen. Consider stable, rational functions $F(s)$, with the property that $F(j\bar{\omega}) = 1$, and $|F(s)| < 1$ for all other complex $s$ with $\text{Re}(s) \geq 0$. By problem 7, you will know these exist. Multiplying $\Delta$ by several of these yields $\Delta_{\text{mod}} \in \mathcal{S}$ such that

$$\|\Delta_{\text{mod}}\|_\infty = |\delta| < 1, \quad \Delta_{\text{mod}}(j\bar{\omega}) = \delta$$
and at all right-half plane poles, \( p \), of \( P \)

\[
|\Delta_{\text{mod}}(p)| < \left| \frac{1}{W_u(p)} \right|
\]

which guarantees no possibility of \( 1 + W_u \Delta_{\text{mod}} \) cancelling any right-half plane poles of \( P \).

Alternatively, we could define an \textit{additive} uncertainty set, for example,

\[
A(P, W_u) := \{ P + W_u \Delta \in \mathcal{R} : \Delta \in \mathcal{R}, \sup_{\omega \in \mathbb{R}} |\Delta(j\omega)| < 1, \#\text{rhpp}(P) = \#\text{rhpp}(P + W_u \Delta) \}
\]

Here, the disks around \( P(j\omega) \) which contain all possible values of \( \tilde{P}(j\omega) \) have radius \( |W_u(j\omega)| \).

Given \( P, C, W_u \) and \( W_p \), two main analysis questions are:

1. \textbf{Robust Stability:} Does \( C \) stabilize every \( \tilde{P} \in M(P, W_u) \)? If so, we say that the pair \( (P, C) \) is robustly stable with respect to multiplicative perturbations defined by \( W_u \).

2. \textbf{Robust Performance:} Suppose the answer to the Robust Stability question is \textbf{yes}. The Robust Performance question is: Is the worst-case performance criterion, defined below as

\[
\mathcal{W}(P, C, W_u, W_p) := \sup_{\tilde{P} \in M(P, W_u)} \left\| \frac{W_p}{1 + \tilde{P}C} \right\|_{\infty}
\]

less than or equal to 1?

Note that for a specific \( \tilde{P} \), the \textit{performance objective} is satisfied if and only if \( |W_p(j\omega)| \leq |1 + \tilde{P}(j\omega)C(j\omega)| \) for all \( \omega \). Graphically, this is equivalent to requiring that the loop gain (complex number) \( \tilde{P}(j\omega)C(j\omega) \) lie outside an open disk of radius \( |W_p(j\omega)| \), centered at \(-1 + j0\). We will see that this is precisely the necessary and sufficient condition for robust performance, and that it can be written as an inequality involving \( P, C, W_p \) and \( W_u \).
4.2 Robust Stability

**Theorem 39** The controller $C$ stabilizes every $\tilde{P} \in M(P,W_u)$ if and only if $C$ stabilizes $P$, and

$$\left\| \frac{W_uPC}{1+PC} \right\|_\infty \leq 1$$

**Remark:** The proof given below assumes that neither $P$ nor $C$ have any imaginary-axis poles. You can rework the proof in those cases if you are interested.

**Proof:** ($\Rightarrow$) Suppose that the condition is violated, that is

$$\left\| \frac{W_uPC}{1+PC} \right\|_\infty > 1.$$  

Choose a particular frequency $\tilde{\omega} \in (0, \infty)$ such that

$$\left| \frac{W_uPC}{1+PC(j\tilde{\omega})} \right| > 1.$$  

(and $j\tilde{\omega}$ is not a pole or zero of $P$ or $C$). For simplicity in notation, all transfer functions are evaluated at $s = j\tilde{\omega}$. Define $\gamma \in \mathbb{C}$ as

$$\gamma := -\frac{1+PC}{C} = PW_u \frac{1+PC}{W_uPC}$$

Note that $|\gamma| < |PW_u|$ by the hypothesis and choice of $\tilde{\omega}$. Construct a plant $\tilde{P} \in M(P,W_u)$ such that $\tilde{P}(j\tilde{\omega}) = P(j\tilde{\omega}) + \gamma$. Hence, at $s = j\tilde{\omega}$, we have

$$1 + \tilde{PC} = 1 + (P + \gamma)C$$

$$= 1 + (P - \frac{1+PC}{C})C$$

$$= 1 + (PC - (1+PC))$$

$$= 0$$

Therefore, the Nyquist Plot of the open-loop gain for the perturbed system passes through the -1 point (at $s = j\tilde{\omega}$), and hence the closed-loop system has a pole at $s = j\tilde{\omega}$. More simply, the rational function

$$\frac{1}{1 + \tilde{PC}}$$

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has a pole at $s = j\bar{\omega}$.

$(\Leftarrow)$ Let $\epsilon \in [0, 1]$, $\Delta \in \mathbb{R}$. Assume that $\sup_{\omega \in \mathbb{R}} |\Delta(j\omega)| < 1$, and that $\#\text{rhpp}(P) = \#\text{rhpp}(P(1 + W_u\Delta))$. Consider the Nyquist plot of

$$P(1 + \epsilon W_u\Delta)C$$

for different values of $\epsilon \in [0, 1]$. At $\epsilon = 0$, this is simply the Nyquist plot of the nominal system, and since we assume that $C$ stabilizes $P$, this must have the correct number of encirclements. The perturbed Nyquist plot is given by $\epsilon = 1$. At $\epsilon = 1$, this is the Nyquist plot for the perturbed system ($\tilde{P}$).

Now, for any $\omega \in \mathbb{R}$, and any $\epsilon \in [0, 1]$, we have

$$|\epsilon W_u\Delta PC| \leq |W_u\Delta PC|$$

$$< |W_uPC|$$

$$\leq |1 + PC|$$

This implies that for all $\epsilon \in [0, 1]$ and all $\omega \in \mathbb{R}$,

$$1 + P(1 + \epsilon W_u\Delta)C = (1 + PC) + (\epsilon W_u\Delta PC)$$

$$\neq 0$$

Hence, as $\epsilon$ ranges from 0 to 1, the Nyquist plot of $P(1 + \epsilon W_u\Delta)C$ **never passes through the $-1$ point**. This means that the number of encirclements of $-1$ by the Nyquist plot of $P(1 + \epsilon W_u\Delta)C$ remains the same for all $\epsilon \in [0, 1]$. We know that

- the number of encirclements of $-1$ by the Nyquist plot of $PC$ equals $\#\text{rhpp}(P) + \#\text{rhpp}(C)$,
- the number of encirclements of $-1$ by the Nyquist plot of $P(1 + W_u\Delta)C$ equals the number of encirclements of $-1$ by the Nyquist plot of $PC$,
- $\#\text{rhpp}(P) = \#\text{rhpp}(P(1 + W_u\Delta))$

Hence, by the Nyquist stability theorem, the controller $C$ stabilizes the plant

$$\tilde{P} = P(1 + W_u\Delta),$$

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as desired. 

This condition can be interpreted graphically. Note that at each \( \omega \), it is necessary that

\[
|1 + P(j\omega)C(j\omega)| \geq |W_u(j\omega)P(j\omega)C(j\omega)|
\]

Geometrically, it means that for every \( \omega \), the point \(-1\) should not be contained in the open disk, centered at \( P(j\omega)C(j\omega) \), with radius \( |W_u(j\omega)P(j\omega)C(j\omega)| \). This disk is simply all of the possible loop gains (at frequency \( \omega \)) that can arise with plants coming from the set \( M(P, W_u) \).

Hence, we would expect that the robust performance condition would require that all possible loop gains remain outside the open disk of radius \( |W_P(j\omega)| \), centered at \(-1 + j0\). This requires that the sum of the radii of these two disks be less than, or equal to the distance between their centers. In terms of the functions, this would be the inequality (for all \( \omega \))

\[
|W_p(j\omega)| + |W_uPC(j\omega)| \leq |1 + PC(j\omega)|
\]

As we see in the next section, this is exactly the correct answer.

### 4.3 Robust Performance

**Theorem 40** Given the four transfer functions \( P, C, W_u, \) and \( W_p \). Assume that \( C \) stabilizes \( P \). Then, \( C \) stabilizes every \( \tilde{P} \in M(P, W_u) \) and

\[
\sup_{\tilde{P} \in M(P, W_u)} \left\| \frac{W_p}{1 + PC} \right\|_\infty \leq 1
\]

if and only if

\[
\sup_{\omega \in \mathbb{R}} \left[ \left| \frac{W_p}{1 + PC}(j\omega) \right| + \left| \frac{W_uPC}{1 + PC}(j\omega) \right| \right] \leq 1
\]

**Proof:** (\( \Rightarrow \)) Suppose that

\[
\sup_{\omega \in \mathbb{R}} \left[ \left| \frac{W_p}{1 + PC}(j\omega) \right| + \left| \frac{W_uPC}{1 + PC}(j\omega) \right| \right] > 1
\]
If the robust stability condition alone is violated, then simply construct a plant \( \tilde{P} \in M(P, W_u) \) such that the perturbed closed-loop is unstable. Otherwise, assume that the robust stability condition is satisfied, but the robust performance condition is not. Choose a frequency \( \bar{\omega} \) such that

\[
|W_p| + |W_u PC| > |1 + PC|
\]

Hence, one can construct an allowable \( \Delta \in \mathcal{R} \), such that at \( \bar{\omega} \)

\[
|W_p| > |1 + PC + \Delta W_u PC|
\]

HOW: For any complex numbers \( a, b \) and \( c \), with \( |a| + |b| > |c| \), choose \( \beta \in \mathbb{C}, |\beta| < 1 \) so that \( a > |c + \beta b| \geq |c| - |b| \). Then \( \left| \frac{a}{c + \beta b} \right| > 1 \). Apply this with \( a = W_p, b = W_u PC, c = 1 + PC, \beta = \Delta \). Then

\[
\left| \frac{W_p}{1 + P (1 + W_u \Delta) C} \right| > 1
\]

so that robust performance is not achieved.

\((\Leftarrow)\) Choose any \( \Delta \in \mathcal{R} \) satisfying the properties in the definition of \( M(P, W_u) \) (so we are picking a \( \tilde{P} \in M(P, W_u) \)). Since the robust stability condition is already satisfied, we are guaranteed that the closed-loop system is stable. At any frequency \( \omega \), we have

\[
|W_p| \leq |1 + PC| - |W_u PC| \leq |1 + PC| - |W_u \Delta PC| \leq |1 + PC + W_u \Delta PC|
\]

which implies that

\[
\left| \frac{W_p}{1 + P (1 + W_u \Delta) C} \right| = \left| \frac{W_p}{1 + \tilde{P} C} \right| \leq 1
\]

for every \( \omega \in \mathbb{R} \). ♦

4.4 References

4.5 Problems

1. TRUE or FALSE (i.e., prove, or give a counterexample): If a single controller \( C \in \mathcal{R} \) stabilizes two plants \( P_1 \) and \( P_2 \), then \( P_1 \) and \( P_2 \) have the same number of poles in \( \mathcal{C}_+ \).

2. Assume that the feedback loop shown below is stable. Let \( T_N \) denote the transfer function from \( d \rightarrow e \).

![Feedback Loop Diagram]

Find \( T_N \) in terms of \( L \), and then invert this, to obtain \( L \) in terms of \( T_N \). Pick \( \alpha > 0 \). At any frequency \( \omega \), show that

\[
|L(j\omega) + 1| < \alpha \quad \Rightarrow \quad |T_N(j\omega)| > \frac{1 - \alpha}{\alpha}
\]

In terms of the Nyquist Stability criterion, what does this imply when \( \alpha \) is very small?

Similarly, choose \( \beta > 1 \). Show that

\[
|T(j\omega)| > \beta \quad \Rightarrow \quad |1 + L(j\omega)| < \frac{1}{\beta - 1}
\]

What does this imply when \( \beta \) is very large?

3. Suppose that the nominal model of the plant

\[
P := \frac{2}{s - 2}
\]

and the uncertainty in the plant model is parametrized by the multiplicative uncertainty discussed in class, with uncertainty weight

\[
W_u := \frac{0.25(0.5s + 1)}{0.03125s + 1}
\]

(a) Draw a Nyquist plot of \( P \). On this Nyquist plot, at the frequencies \( \omega = 0.1, 0.18, 0.31, 0.56, 1, 1.8, 3.1, 5.6, 10 \), draw disk of correct radius which show the possible values that \( \tilde{P}(j\omega) \) can take on, due to the uncertainty description.
(b) Find the largest \( \bar{\tau} \) (using numerical or hand calculation – whichever you want) such that
\[
\left\{ P \frac{1}{\tau s + 1} : \tau < \bar{\tau} \right\} \subset M(P, W_u)
\]

(c) Find extreme values \( \delta \) and \( \bar{\delta} \) such that
\[
\left\{ \frac{1 + \delta}{s - 1 - \delta} : \delta \leq \delta \leq \bar{\delta} \right\} \subset M(P, W_u)
\]

(d) Find the largest \( \bar{\tau} \) such that
\[
\left\{ P \frac{-\tau s + 1}{\tau s + 1} : \tau \in (0, \bar{\tau}] \right\} \subset M(P, W_u)
\]

(e) Find extreme values \( \xi \) and \( \bar{\xi} \) such that
\[
\left\{ P \frac{70^2}{s^2 + 2\xi 70s + 70^2} : \xi \leq \xi \leq \bar{\xi} \right\} \subset M(P, W_u)
\]

(f) Find the largest integer \( \bar{m} > 0 \) such that
\[
\left\{ P \left( \frac{1}{0.01s + 1} \right)^m : m = 0, 1, \ldots, \bar{m} \right\} \subset M(P, W_u)
\]

(g) Consider the performance weighting function, parametrized by a parameter \( \alpha \).
\[
W_p(s) = \frac{0.5s + 50\alpha}{s + \alpha}
\]

What type of performance objective is implied by this weighting function? Using the results from problem 8, determine the ranges for gain crossover for a few values of \( \alpha \). Also determine the restrictions on the phase at crossover. Use these bounds, as well as the simple necessary condition for robust performance that for all \( \omega \in \mathbb{R} \), we must have
\[
\min \{|W_u(j\omega)|, |W_p(j\omega)|\} \leq 1
\]
to get a rough idea on what values of \( \alpha \) is robust performance achievable.
(h) By absolutely whatever technique you like, maximize the value of \( \alpha \) for which you can design a controller \( C \) that achieves robust performance (using \( W_p \) as performance objective) for the set \( M(P,W_u) \). In section ??, and the associated problems, we have developed a convex minimization to approximately solve this problem.

(i) Use all of the extreme plant models developed in parts 3b-3f, and compute (simulate) step responses of the closed-loop system (either step disturbances at output of plant, or step reference inputs for tracking). Also simulate the closed-loop response using the “worst-case” plant from the set \( M(P,W_u) \).

4. Given a positive \( \bar{\omega} > 0 \), and a complex number \( \delta \), with \( \text{Imag} (\delta) \neq 0 \), show (ie. give explicit constructive procedure – how to get \( \beta \) from \( \omega \) and \( \delta \)) that there is a \( \beta > 0 \) such that by proper choice of sign

\[
\pm |\delta| \left| \frac{s - \beta}{s + \beta} \right|_{s = j\bar{\omega}} = \delta
\]

Let \( H(s) \) denote this transfer function. Sketch a Bode plot, clearly marking \( \beta \) on the frequency axis, and compute

\[
\max_{\omega \in \mathbb{R}} |H(j\omega)|
\]

5. This is a similar, but different approach to the robust stability problem we have considered in Section 4.2 of the notes. Suppose that \( M \) is a stable SISO system with transfer function \( M(s) \). Introduce the notation

\[
\|M\|_\infty := \max_\omega |M(j\omega)|
\]

For simplicity, assume that \( M \) is not identically zero, and also assume that the peak is achieved at some finite frequency (rather than as \( \omega \to \infty \)).

(a) Using a Nyquist argument, show that the feedback loop
is stable for all stable systems \( \Delta \), satisfying

\[
\|\Delta\|_\infty < \frac{1}{\|M\|_\infty}
\]

(b) Show that there is a stable, 1st order, or 0th order transfer function \( \Delta \) satisfying

\[
\|\Delta\|_\infty = \frac{1}{\|M\|_\infty}
\]

and which causes the feedback loop shown below to be unstable

\[\begin{array}{c}
\text{r}_1 \\
\text{e}_1 \\
\text{M} \\
\text{e}_2 \\
\text{r}_2
\end{array}\]

\[\begin{array}{c}
\text{y}_1 \\
\text{y}_2 \\
\Delta
\end{array}\]

(c) Finally, suppose that \( W(s) \) is a stable, minimum-phase transfer function, with \( W(\omega = \infty) \neq 0 \) (hence, the numerator order is equal to the denominator order). Show that the feedback loop is stable for all stable transfer functions \( \Delta \) satisfying

\[
|\Delta(j\omega)| \leq |W(j\omega)| \quad \forall \omega
\]

if and only if \( \|WM\|_\infty < 1 \).

6. Suppose that \( M(s) \in \mathcal{S} \), and \( \|M\|_\infty < 1 \). Show that \( H(s) := 1 \pm M(s) \in \mathcal{U}_S \).

7. For \( \xi > 0 \), and \( \omega_n \in \mathbb{R} \), define

\[
\hat{g}(s) := \frac{2\xi \omega_n s}{s^2 + 2\xi \omega_n s + \omega_n^2}
\]

(a) Calculate \( \|\hat{g}\|_\infty \)

(b) Find a real number \( \omega_p \) such that

\[
|\hat{g}(j\omega_p)| = \|\hat{g}\|_\infty
\]

(c) What is \( \hat{g}(s)\big|_{s=j\omega_p} \).

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8. In our standard SISO Robust Performance problem, suppose that $\omega_C$ is a gain-crossover frequency,

$$|L(j\omega_C)| := |PC(j\omega_C)| = 1$$

(a) Show that if $\omega_C$ is a crossover frequency, and Robust performance is achieved, then it **must** be that

$$|W_u(j\omega_C)| + |W_p(j\omega_C)| \leq 2$$

(b) If the controller $C$ achieves Robust Performance, show that the phase angle at crossover must satisfy

$$\cos(\angle L(j\omega_C)) > \frac{1}{2}(|W_u(j\omega_C)| + |W_p(j\omega_C)|)^2 - 1$$

9. Suppose that $P = \frac{1}{s^2+4}$, and $W_u \in \mathcal{S}$, say $W_u = \frac{5s+2}{s+1}$. Suppose that $\tilde{P} \in M(P,W_u)$ has poles at $\pm j\alpha$, for some real number $\alpha$. What can you determine about $\alpha$.

10. In parts of this problem, you may need the following fact (the maximum modulus theorem, and can be proven using a Nyquist-based proof). If $G \in \mathcal{S}$, then

$$\|G\|_\infty := \max_{\omega \in \mathbb{R}} |G(j\omega)| = \max_{s \in \mathbb{C}, \text{Re}(s) \geq 0} |G(s)|$$

This also holds for column vectors - if $G_1, G_2 \in \mathcal{S}$, then

$$\left\| \begin{array}{c} G_1 \\ G_2 \end{array} \right\|_\infty := \max_{\omega \in \mathbb{R}} \left\| \begin{array}{c} G_1(j\omega) \\ G_2(j\omega) \end{array} \right\|_2 = \max_{s \in \mathbb{C}, \text{Re}(s) \geq 0} \left\| \begin{array}{c} G_1(s) \\ G_2(s) \end{array} \right\|_2$$

Obviously, it holds for row vectors as well (using the Euclidean norm).

(a) Preliminary: **True/False**: Given $u \in \mathbb{C}^n, v \in \mathbb{C}^n, \beta > 0$. If $\|u\|_2 < \beta$, and $\|\beta v\|_2 \leq 1$, then $|v^*u| < 1$.

(b) Let $(N, D)$ be a coprime factorization of $P \in \mathcal{R}$. Suppose $R \in \mathcal{S}$, with $|R(\infty)| < |D(\infty)|$.

$$F(N, D, R) := \left\{ \begin{array}{c} \tilde{N}(s) \\ \tilde{D}(s) \end{array} : \left\| \begin{array}{c} N(s) - \tilde{N}(s) \\ D(s) - \tilde{D}(s) \end{array} \right\|_2 < |R(s)| \ \forall \text{Re}(s) \geq 0, \text{ including } s = \infty \right\}$$
Take $P$ as in problem 9, with any coprime factorization $(N, D)$ you choose. Define $R := 0.3$. Show that there are $\tilde{P} \in F(N, D, R)$ with
   i. 2 right-half plane poles, with real part greater than 0
   ii. 2 imaginary axis poles, but not at $\pm j2$
   iii. no poles in the right-half plane

(c) Back to general $R \in \mathcal{S}$. Let $C \in \mathcal{R}$ stabilize $P$. Let $(N_C, D_C)$ be a coprime factorization of $C$ such that $N_C N + D_C D = 1$. Show that $C$ stabilizes every $\tilde{P} \in F(N, D, R)$ if
   $$\|R [D_C \ N_C]\|_\infty \leq 1 \quad (4.10)$$
   (actually if and only if, but you only need to show if). **Hint:** Use (10a) and problem 6.

(d) Parametrize all pairs $(N_c, D_c)$ in $\mathcal{S}$ such that $N_c N + D_c D = 1$. Show that the left-hand side of (4.10) is a convex function of the free parameter