Output Feedback Stabilization problem

Given matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_u}$, $C \in \mathbb{R}^{n_y \times n}$.

Define $n$’th order, linear system $P$

$$
\dot{x}(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t)
$$

Goal: find all (if any) finite dimensional, linear feedback controllers, $K$

$$
\dot{\eta}(t) = \bar{A}\eta(t) + \bar{B}y(t) \\
u(t) = \bar{C}\eta(t) + \bar{D}y(t)
$$

such that the closed-loop system

$$
\begin{bmatrix}
\dot{x} \\
\dot{\eta}
\end{bmatrix} = 
\begin{bmatrix}
A + B\bar{D}C & B\bar{C} \\
\bar{B}C & \bar{A}
\end{bmatrix}
\begin{bmatrix}
x \\
\eta
\end{bmatrix}.
$$
Internal Stabilization

More generally, consider \( n \)’th order, linear system \( G \) with two types of inputs and two types of outputs,

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B_1 d(t) + B_2 u(t) \\
e(t) &= C_1 x(t) + D_{11} d(t) + D_{12} u(t) \\
y(t) &= C_2 x(t) + D_{21} d(t)
\end{align*}
\]

Goal: find all (if any) finite dimensional, linear feedback controllers, \( K \)

\[
\begin{align*}
\dot{\eta}(t) &= \bar{A}\eta(t) + \bar{B} y(t) \\
u(t) &= \bar{C}\eta(t) + \bar{D} y(t)
\end{align*}
\]

such that the closed-loop system

is internally stable. Easy to check that closed-loop dynamics are

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{\eta}(t) \\
e(t)
\end{bmatrix} =
\begin{bmatrix}
A + B_2 \bar{D} C_2 & B_2 \bar{C} & B_1 + B_2 \bar{D} D_{21} \\
\bar{B} C_2 & \bar{A} & \bar{B} D_{21} \\
C_1 + D_{12} \bar{D} C_2 & D_{12} \bar{C} & D_{11} + D_{12} \bar{D} D_{21}
\end{bmatrix}
\begin{bmatrix}
x(t) \\
\eta(t) \\
d(t)
\end{bmatrix}
\]

and internal stability of closed-loop system is governed by the eigenvalues of

\[
\begin{bmatrix}
A + B_2 \bar{D} C_2 & B_2 \bar{C} \\
\bar{B} C_2 & \bar{A}
\end{bmatrix}
\]
Stabilization by Dynamic Output feedback

In the first case (only inputs/outputs are controls/measurements), the closed-loop “A” matrix decomposes as

\[
\begin{bmatrix}
A + B\tilde{D}C & B\tilde{C} \\
\tilde{B}C & \tilde{A}
\end{bmatrix}
= 
\begin{bmatrix}
A & 0 \\
0 & 0_m
\end{bmatrix}
+ 
\begin{bmatrix}
B & 0 \\
0 & I_m
\end{bmatrix}
\begin{bmatrix}
\tilde{D} & \tilde{C} \\
\tilde{B} & \tilde{A}
\end{bmatrix}
\begin{bmatrix}
C & 0 \\
0 & I_m
\end{bmatrix}
\]

In the second case, with two types of inputs/outputs, the closed-loop “A” matrix decomposes as

\[
\begin{bmatrix}
A + B_2\tilde{D}C_2 & B_2\tilde{C} \\
\tilde{B}C_2 & \tilde{A}
\end{bmatrix}
= 
\begin{bmatrix}
A & 0 \\
0 & 0_m
\end{bmatrix}
+ 
\begin{bmatrix}
B_2 & 0 \\
0 & I_m
\end{bmatrix}
\begin{bmatrix}
\tilde{D} & \tilde{C} \\
\tilde{B} & \tilde{A}
\end{bmatrix}
\begin{bmatrix}
C_2 & 0 \\
0 & I_m
\end{bmatrix}
\]

which is structurally identical, with \(B_2\) replacing by \(B\), and so on.

Since they are the “same,” use the simpler notation, and consider the first case to derive all stabilizing controllers.

The dynamic, output-feedback stabilization problem is: Given the matrices \(A\), \(B\), and \(C\), find, if they exist, an integer \(m \geq 0\), and a matrix \(M \in \mathbb{R}^{(n_u+m)\times(n_y+m)}\) such that

\[
\begin{bmatrix}
A & 0 \\
0 & 0_m
\end{bmatrix}
+ 
\begin{bmatrix}
B & 0 \\
0 & I_m
\end{bmatrix}
M
\begin{bmatrix}
C & 0 \\
0 & I_m
\end{bmatrix}
\]

is Hurwitz.
Stabilizability

Definition 1 The pair of matrices \((A, B)\) is \textbf{stabilizable} if there a matrix 
\(F \in \mathbb{R}^{n_u \times n}\) such that \(A + BF\) is Hurwitz (i.e., all eigenvalues have negative 
real part). Such a matrix \(F\) will be referred to as a \textbf{stabilizing state} 
feedback for the pair \((A, B)\).

Remark 1: The non-dynamic state-feedback \(u(t) = Fx(t)\) stabilizes the system 
\[ \dot{x}(t) = Ax(t) + Bu(t). \]

Remark 2: Recall that \(A \in \mathbb{R}^{n \times n}\) is Hurwitz if and only if there exists 
\(P \in \mathbb{R}^{n \times n}, P = P^T > 0\), such that \(A^T P + PA < 0\). Hence, an equivalent 
statement of stabilizability of \((A, B)\) is: there exists \(P_F \in \mathbb{R}^{n \times n}, P_F = 
P_F^T > 0\), and a matrix \(F \in \mathbb{R}^{n_u \times n}\) such that 
\[ (A + BF)^T P_F + P_F (A + BF) < 0. \] \hspace{1cm} (1.2)

Remark 3: Another equivalent characterization of stabilizability: The matrix 
\([A - \lambda I \ B]\) has full row rank for all \(\lambda \in \mathbb{C}\), with \(\text{Re}(\lambda) \geq 0\).
Theorem 2 The pair \((A, B)\) is stabilizable if and only if there exist \(W \in \mathbb{R}^{n \times n}\) and \(R \in \mathbb{R}^{n_u \times n}\) such that \(W = W^T > 0\), and
\[
AW + WA^T + BR + R^T B^T < 0.
\] (1.3)

Proof:

\(\rightarrow\) By assumption, there exist matrices \(F \in \mathbb{R}^{n_u \times n}\) and \(P_F \in \mathbb{R}^{n \times n}, P_F = P_F^T > 0\) such that
\[
(A + BF)^T P_F + P_F (A + BF) < 0.
\]
Define \(W := P_F^{-1}\). Note that \(W \in \mathbb{R}^{n \times n}, W = W^T > 0\). Also define \(R := FW\). Then
\[
AW + WA^T + BR + R^T B^T < 0.
\]

\(\leftarrow\) The argument given above is reversed. \textcircled{12}
Theorem 3 Let \( m \geq 0 \). Define matrices \( A_m^e \) and \( B_m^e \) as

\[
A_m^e := \begin{bmatrix} A & 0 \\ 0 & 0_m \end{bmatrix}, \quad B_m^e := \begin{bmatrix} B & 0 \\ 0 & I_m \end{bmatrix}
\]

Then \((A, B)\) is stabilizable if and only if \((A_m^e, B_m^e)\) is stabilizable.

Proof

← Using Theorem 2, there must exist \( 0 \prec W^e \in \mathbb{R}^{(n+m) \times (n+m)} \), and \( R^e \in \mathbb{R}^{(n_u+m) \times (n+m)} \),

\[
W^e = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^T & W_{22} \end{bmatrix}, \quad R^e = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}
\]

such that

\[
A_m^e W^e + W^e A_m^e T + B_m^e R^e + R^e T B_m^e T \prec 0. \tag{1.4}
\]

Note \( W_{11} = W_{11}^T \succ 0 \). Obviously, the \((1, 1)\) block of equation 1.4 is negative definite, specifically,

\[
AW_{11} + W_{11} A^T + BR_{11} + R_{11} B^T \prec 0.
\]

Using Theorem 2, this means that the pair \((A, B)\) is stabilizable.

→ Let \( F \) be such that \( A + BF \) is Hurwitz. Define

\[
F_m^e := \begin{bmatrix} F & 0 \\ 0 & -I_m \end{bmatrix}
\]

Then

\[
A_m^e + B_m^e F_m^e = \begin{bmatrix} A + BF & 0 \\ 0 & -I_m \end{bmatrix}
\]

which is clearly Hurwitz, as desired. 

Detectability

The properties for detectability are similar to those of stabilizability.

Definition 4 The pair of matrices \((A, C)\) is detectable if there exists a matrix \(L \in \mathbb{R}^{n \times n_y}\) such that \(A + LC\) is Hurwitz.

Remark: In this situation, the observer (with non-dynamic processing of the residual \(y - C\hat{x}\))

\[
\dot{\hat{x}} = A\hat{x} - L(y - C\hat{x})
\]

yields a exponentially stable estimate of the state \(x\) for the system

\[
\begin{align*}
\dot{x} &= Ax \\
y &= Cx.
\end{align*}
\]

Theorem 5 The pair \((A, C)\) is detectable if and only if there exist \(P \in \mathbb{R}^{n \times n}, P = P^T \succ 0\), and a matrix \(H \in \mathbb{R}^{n \times n_y}\) such that

\[
A^T P + PA + HC + C^T H^T \prec 0 \tag{1.5}
\]

As in the state-feedback problem, dynamic extension does not make the observation problem any easier.

Theorem 6 Let \(m\) be any nonnegative integer. Define matrices \(A^e_m\) and \(C^e_m\) as

\[
A^e_m := \begin{bmatrix} A & 0 \\ 0 & 0_m \end{bmatrix}, \quad C^e_m := \begin{bmatrix} C & 0 \\ 0 & I_m \end{bmatrix}
\]

Then \((A, C)\) is detectable if and only if \((A^e_m, C^e_m)\) is detectable.
Theorem 7  Given open-loop state space data $A$, $B$, and $C$. There exist an integer $m \geq 0$, and a matrix $M \in \mathbb{R}^{(n_u+m) \times (n_y+m)}$ such that $(A^e_m + B^e_mMC^e_m)$ is Hurwitz if and only if

1. the pair $(A, B)$ is stabilizable, and
2. the pair $(A, C)$ is detectable.

Furthermore, if these two conditions hold, then there exist

1. $F \in \mathbb{R}^{n_u \times n}$ such that $(A + BF)$ is Hurwitz
2. $L \in \mathbb{R}^{n \times n_y}$ such that $(A + LC)$ is Hurwitz

and

$$
\begin{bmatrix}
\dot{\eta} \\
u
\end{bmatrix} =
\begin{bmatrix}
A + BF + LC & -L \\
F & 0
\end{bmatrix}
\begin{bmatrix}
\eta \\
y
\end{bmatrix} =
\begin{bmatrix}
\tilde{A} & \tilde{B} \\
\tilde{C} & \tilde{D}
\end{bmatrix}
\begin{bmatrix}
\eta \\
y
\end{bmatrix}
$$

is a exponentially stabilizing, output-feedback controller.
Proof

→ Viewing the Hurwitz matrix $A^e_m + B^e_mMC^e_m$ as

\[
A^e_m + B^e_mMC^e_m
\]

it is clear that the pair $(A^e_m, B^e_m)$ is stabilizable, and the pair $(A^e_m, C^e_m)$ is detectable. Hence, by Theorems 3 and 6, the pair $(A, B)$ is stabilizable, and the pair $(A, C)$ is detectable.

← Take $\bar{D} := 0, \bar{C} := F, \bar{B} := -L$ and $\bar{A} := A + BF + LC$, and define

\[
M := \begin{bmatrix}
\bar{D} & \bar{C} \\
\bar{B} & \bar{A}
\end{bmatrix}.
\]

With this state-space realization for a feedback controller, the closed-loop dynamics are governed by

\[
A_{clp} := A^e_m + B^e_mMC^e_m = \begin{bmatrix}
A & BF \\
-LC & A + BF + LC
\end{bmatrix}.
\]

Define an invertible matrix $T \in \mathbb{R}^{2n \times 2n}$ as

\[
T := \begin{bmatrix}
I_n & -I_n \\
0 & I_n
\end{bmatrix}
\]

Note that

\[
TA_{clp}T^{-1} = \begin{bmatrix}
A + LC & 0 \\
-LC & A + BF
\end{bmatrix},
\]

which is clearly Hurwitz. \#.
all stabilizing controllers

**Theorem 8** Let $A, B, C$ be the open-loop state-space data. Assume $(A, B)$ is stabilizable and $(A, C)$ is detectable. So, choose matrices $F \in \mathbb{R}^{n_u \times n}$, and $L \in \mathbb{R}^{n \times n_y}$ such that $A + BF$ and $A + LC$ are Hurwitz.

For every internally stabilizing controller $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$, there exists a stable system $(A_Q, B_Q, C_Q, D_Q)$ such that a realization of the controller, possibly with stable, yet uncontrollable and/or unobservable modes is:

$$
\begin{bmatrix}
\dot{\eta}_o \\
\dot{\eta}_Q \\
u
\end{bmatrix} =
\begin{bmatrix}
A + BF + LC + BD_QC & BC_Q & -L - BD_Q \\
B Q C & A_Q & -B_Q \\
F + D_Q C & C_Q & -D_Q
\end{bmatrix}
\begin{bmatrix}
\eta_o \\
\eta_Q \\
y
\end{bmatrix}
$$

(1.6)

Moreover, for every choice of stable system $(A_Q, B_Q, C_Q, D_Q)$, the formula above is a stabilizing controller.

A block diagram of this controller structure is shown below:
Suppose $A_Q, B_Q, C_Q$ and $D_Q$ are matrices of appropriate dimensions, with $A_Q$ Hurwitz. Implement the state-space parametrization of the controller given above in equation 1.6 (as in the figure above). The closed-loop $A$ matrix, $A_{clp}$, is

$$
A_{clp} = \begin{bmatrix}
A - BD_Q C & B (F + D_Q C) & BC_Q \\
-(L + BD_Q) C & A + BF + LC + BD_Q C & BC_Q \\
-B_Q C & B_Q C & A_Q
\end{bmatrix}
$$

Define an invertible matrix $T$ as

$$
T := \begin{bmatrix}
I_n & -I_n & 0 \\
0 & 0 & I_{nQ} \\
0 & I_n & 0
\end{bmatrix}
$$

Then,

$$
TA_{clp}T^{-1} = \begin{bmatrix}
A + LC & 0 & 0 \\
-B_Q C & A_Q & 0 \\
-(L + BD_Q) C & BC_Q & A + BF
\end{bmatrix}
$$

By assumption, each of the blocks on the diagonal are Hurwitz, so the closed-loop matrix $A_{clp}$ is Hurwitz as well.
Proof

Suppose that $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is a realization of a exponentially stabilizing controller, with state dimension $m \geq 0$. Then, the matrix

$$A_m^e + B_m^e \begin{bmatrix} \bar{D} & \bar{C} \\ \bar{B} & \bar{A} \end{bmatrix} C_m^e$$

is Hurwitz, and define $A_Q$ to be it. Define $D_Q := -\bar{D}$, and

$$B_Q := \begin{bmatrix} L - B\bar{D} \\ -\bar{B} \end{bmatrix}, C_Q := \begin{bmatrix} -F + \bar{D}C & \bar{C} \end{bmatrix}, T := \begin{bmatrix} I_n & -I_n & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_m \end{bmatrix}$$

Plugging in to the (claimed) parametrization gives

$$\begin{bmatrix} \dot{\eta}_0 \\ \dot{\eta}_{Q_1} \\ \dot{\eta}_{Q_2} \\ u \end{bmatrix} = \begin{bmatrix} A + BF + LC - B\bar{D}C & B(-F + \bar{D}C) & B\bar{C} & -L + B\bar{D} \\ (L - B\bar{D})C & A + B\bar{D}C & B\bar{C} & -L + B\bar{D} \\ -\bar{B}C & \bar{B}C & \bar{A} & \bar{B} \\ F - \bar{D}C & -F + \bar{D}C & \bar{C} & \bar{D} \end{bmatrix} \begin{bmatrix} \eta_0 \\ \eta_{Q_1} \\ \eta_{Q_2} \\ y \end{bmatrix}$$

After applying the coordinate transformation $T$, the state equations become

$$\begin{bmatrix} \dot{\eta}_0 - \dot{\eta}_{Q_1} \\ \dot{\eta}_{Q_1} \\ \dot{\eta}_{Q_2} \\ u \end{bmatrix} = \begin{bmatrix} A + BF & 0 & 0 & 0 \\ (L - B\bar{D})C & A + LC & B\bar{C} & -L + B\bar{D} \\ -\bar{B}C & 0 & \bar{A} & \bar{B} \\ F - \bar{D}C & 0 & \bar{C} & \bar{D} \end{bmatrix} \begin{bmatrix} \eta_0 - \eta_{Q_1} \\ \eta_{Q_1} \\ \eta_{Q_2} \\ y \end{bmatrix}$$

The first states are exp. stable, and uncontrollable. Eliminate, leaving

$$\begin{bmatrix} \dot{\eta}_{Q_1} \\ \dot{\eta}_{Q_2} \\ u \end{bmatrix} = \begin{bmatrix} A + LC & B\bar{C} & -L + B\bar{D} \\ 0 & \bar{A} & \bar{B} \\ 0 & \bar{C} & \bar{D} \end{bmatrix} \begin{bmatrix} \eta_{Q_1} \\ \eta_{Q_2} \\ y \end{bmatrix}$$

(1.7)
The first states are exp. stable, and unobservable. Eliminate, leaving

$$\begin{bmatrix} \dot{\eta}_{Q_2} \\ u \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} \begin{bmatrix} \eta_{Q_2} \\ y \end{bmatrix}$$

which is the exponentially stabilizing controller $K$, as claimed. ♦
Consider the plant $G$ with two types of inputs and two types of outputs,

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B_1d(t) + B_2u(t) \\
e(t) &= C_1x(t) + D_{11}d(t) + D_{12}u(t) \\
y(t) &= C_2x(t) + D_{21}d(t)
\end{align*}
\]

Base stabilization on the triple $(A, B_2, C_2)$. Erase $Q$ from the parameterization, introduce two signals, $w$ and $z$, and define $J$ as

\[
\begin{align*}
\dot{\eta}_o(t) &= (A + B_2F + LC_2)\eta_o(t) - Ly(t) + B_2w(t) \\
u(t) &= F\eta_o(t) + w(t) \\
z(t) &= C_2\eta_o(t) - y(t)
\end{align*}
\]
Verify that $T$, above right, is

$$
T = \begin{bmatrix}
A & B_2 F & B_1 & B_2 \\
-LC_2 & A + B_2 F + LC_2 & -LD_{21} & B_2 \\
C_1 & D_{12} F & D_{11} & D_{12} \\
-C_2 & C_2 & -D_{21} & 0
\end{bmatrix}
$$

Moreover, the $(2,2)$ entry of the operator $T$ is $0$,

$$
T_{22} = \begin{bmatrix}
A & B_2 F & B_2 \\
-LC_2 & A + B_2 F + LC_2 & -LD_{21} & B_2 \\
-C_2 & C_2 & 0 \\
A + LC_2 & 0 & 0 \\
-LC_2 & A + B_2 F & B_2 \\
-C_2 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} = 0.
$$

The closed-loop operator from $d$ to $e$ involves the free parameter $Q$ in feedback around the “bottom” of $T$.

The $(2,2)$ entry of $T$ is zero, hence, for all exponentially stabilizing, FDLTI
controllers, the zero-state, closed-loop operator from \( d \) to \( e \) is an affine function of the free parameter \( Q \).
The history of this parameterization is long, and covers many interesting periods. The most well-known references are:

Youla, Jabr, Bongiorno (1976, vol. 21, page 319-338, IEEE Transactions on Automatic Control);


The presentation in this section is slightly different, but the main ideas can be found in

