Introduction

In this report, we discuss our progress on nonlinear systems analysis, based on Lyapunov and dissipation inequalities, using sum-of-squares (SOS) decompositions to verify set containments.

In the past year, we have improved our ability to handle uncertain system dynamics, including parametric and unmodeled dynamics. We continue to use simulation as a key step in aiding the nonconvex search for proofs (Lyapunov functions) and proof certificates (multipliers). We also began more detailed study of systems with marginally stable linearizations (ie., adaptive systems). Finally, we make precise our claim that these techniques represent a definitive improvement over linearized analysis.

**Notation:** $\mathbb{R}[x]$ represents the set of polynomials in $x$ with real coefficients. For $\pi \in \mathbb{R}[x]$, $\partial(\pi)$ denotes the degree of $\pi$. The subset $\Sigma[x] := \{\pi_1^2 + \cdots + \pi_m^2 : \pi_1, \cdots, \pi_m \in \mathbb{R}[x]\}$ is the set of SOS polynomials.

**Uncertain Systems Analysis**

In previous years reports, uncertainty in the vector field was limited to parameter uncertainty, and specifically to affine dependence on a parameter vector $\delta$ lying in a polytope $\Delta$, namely

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} \delta_i f_i(x(t)), \quad \delta \in \Delta$$

In this report, we extend the uncertainty model to include non-affine dependence

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} \delta_i f_i(x(t)) + \sum_{j=1}^{m_{pu}} g_j(\delta)f_{m+j}(x(t))$$
and even unmodeled dynamics,
\[
\dot{x}(t) = f_0(x(t), w(t)) + \sum_{i=1}^{m} \delta_i f_i(x(t), w(t)) + \sum_{j=1}^{n} g_j(\delta) f_{m+j}(x(t), w(t))
\]
\[
z = h(x)
\]
\[
w = \Phi(z)
\]
Here \(\Phi\) represents unmodeled dynamics, for example, finite-dimensional, linear, time-invariant operators with specified upper bound on induced \(L_2\) norm, for example \(\|\Phi\|_{L_2 \rightarrow L_2} < 1\).

The tools we have developed to address this problem are

1. region-of-attraction analysis for systems with affine parameter uncertainty using a single Lyapunov function
2. local induced \(L_2 \rightarrow L_2\) gain analysis for systems with affine parameter uncertainty, using a single polynomial storage function
3. covering, with a polytope, the graph of vector-valued polynomial function over a polytopic domain,
4. informal branch-and-bound
5. local small gain theorems.

Next, we illustrate the calculations that are possible with these methods.

**Controlled short period aircraft dynamics, parametric uncertainty**

We apply the robust ROA analysis for uncertain controlled short period aircraft dynamics (see the project website for the parameters used in the model)

\[
\dot{x}_p = \begin{bmatrix}
c_{01}(x_p) + \delta_1 c_{11}(x_p) + \delta_1^2 q_{33}(x_p) \\
q_{02}(x_p) + \delta_1 \ell_{12} x_p + \delta_2 q_{22}(x_p)
\end{bmatrix} + \begin{bmatrix}
\ell_1^T x_p + b_{11} + b_{12} \delta_1 \\
b_{21} + b_{22} \delta_2
\end{bmatrix} u,
\]

where \(x_p = [x_1 \ x_2 \ x_3]^T\), \(x_1\), \(x_2\), and \(x_3\) denote the pitch rate, the angle of attack, and the pitch angle, respectively, \(c_{01}\) and \(c_{11}\) are cubic polynomials, \(q_{02}\), \(q_{22}\), and \(q_{33}\) are quadratic polynomials, \(\ell_{12}\) and \(\ell_6\) are vectors in \(R^3\), \(b_{11}\), \(b_{12}\), \(b_{21}\), and \(b_{22}\) \(\in R\), and \(u\), the elevator deflection, is the control input. Variations in the center of gravity in the longitudinal direction are modeled by \(\delta_1 \in [0.99,2.05]\) and variations in the mass are modeled \(\delta_2 \in [-0.1,0.1]\). Note that the parametric uncertainty includes one nonaffine term (ie., \(\delta_1^2\)). The control input is determined by \(\dot{x}_4 = -0.864 y_1 - 0.321 y_2\) and \(u = 2 x_4\), where \(x_4\) is the controller state and the plant output \(y = [x_1 \ x_3]^T\). Define \(x := [x_p^T \ x_4]^T\) and the shape factor \(p(x) := x^T x\).

We applied a branch-and-bound type procedure with \(\partial(V) = 2\) and \(\partial(V) = 4\) on a 9-processor computer cluster: after the first \(B&B\) iteration, the cell with the smallest lower bound is subdivided into 3 subcells and cells with 2-nd, 3-rd, and 4-th smallest lower bounds are sub-divided into 2 subcells. Fig. 1 shows the lower bounds and upper bounds. Note that quadratic Lyapunov functions (several, as different Lyapunov functions are employed
in different cells across the parameter space) certify that all initial conditions \(x_0 \in \mathbb{R}^4\) satisfying \(x_0^T x_0 \leq 5.4\) are in the region-of-attraction. Likewise, a collection of quartic Lyapunov functions certify that all initial conditions \(x_0 \in \mathbb{R}^4\) satisfying \(x_0^T x_0 \leq 7.8\) are in the region-of-attraction. The smallest value of \(p\) attained on divergent trajectories, \(\beta^\text{nc}\), is 8.6 and obtained for \((\delta_1, \delta_2) = (2.039, -0.099)\) and the initial condition \((0.17, 2.65, -0.10, 1.24)\).

**Aircraft dynamics, parametric uncertainty and unmodeled dynamics**

Next, consider the same system with additional unmodeled dynamics at the plant input, as shown in Figure 2.

Figure 2: Controlled short period aircraft dynamics with unmodeled dynamics \((\delta_p := (\delta_1, \delta_2))\).

The assumption is that \(\Phi\) is any stable, linear time-invariant (this could be relaxed) operator, with induced \(L_2\) norm less than 1. We again repeat the analysis using both quadratic and quartic storage functions, which locally certify bounds on the gain (with \(\Phi\) removed) from \(w\) to \(z\) (recall that this system is not globally stable) in the presence of the parametric uncertainty. These local \(L_2\) gains are used in conjunction with a local small-gain theorem to yield results such as:

- (using quadratic storage functions) For all (finite-dimensional, linear, time-invariant)
Φ, satisfying \( \| \Phi \|_{L_2 \rightarrow L_2} \leq 1 \), assuming that the initial condition of Φ is 0, then all plant/controller initial conditions \( x_0 \in \mathbb{R}^4 \) satisfying \( x_0^T x_0 \leq 2.4 \), are in the robust region-of-attraction.

- (using quartic storage functions) For all (finite-dimensional, linear, time-invariant) \( \Phi \), satisfying \( \| \Phi \|_{L_2 \rightarrow L_2} \leq 1 \), assuming that the initial condition of Φ is 0, then all plant/controller initial conditions \( x_0 \in \mathbb{R}^4 \) satisfying \( x_0^T x_0 \leq 4.1 \), are in the robust region-of-attraction.

In conclusion, we have established provable and certifiable inner estimates of the region-of-attraction of a (nominally) 4-state nonlinear system with both parametric and unmodeled dynamics uncertainty. The informal use of branch-and-bound in the parametric uncertainty space was handled efficiently using a small-scale parallel cluster of 9 machines.

**Viewing our approach as quantitative extension of linearized analysis**

Practical nonlinear analysis often couples extensive nonlinear simulation with extensive linearized analysis (such as stability, stability margins, Bode plots of linearized I/O maps, etc). Here we show that common linearized analysis techniques can be rigorously quantified using the SOS approaches. The next lemma is key to these derivations.

**Lemma:** Let \( z(x) \) be a vector of all monomials of degree 2 with no repetition. Let \( Q = Q^T \succ 0 \). There exists a positive definite matrix \( H = H^T \) such that \( x^T xx^T Qx = z(x)^T Hz(x) \).

**SOS region-of-attraction analysis:** For the autonomous system \( \dot{x} = f(x) \), and a positive-definite function \( p \), if there exist positive-definite, radially unbounded function \( l_1 \), positive-definite function \( l_2 \), SOS polynomials \( s_1, s_2 \) and \( s_3 \), a polynomial function \( V \), and positive constants \( \gamma \) and \( \beta \) such that

\[
\begin{align*}
V(0) &= 0 \quad &\text{(1a)} \\
-[(\beta - p)s_1 + (V - \gamma)] &\in \Sigma[x], \quad &\text{(1b)} \\
-[(\gamma - V)s_2 + \nabla V f s_3 + l_2] &\in \Sigma[x], \quad &\text{(1c)}
\end{align*}
\]

then \( \{x : p(x) \leq \beta \} \subseteq \{x : V(x) \leq \gamma \} =: \Omega \), and for all \( x(0) \in \Omega \), the solution satisfies \( x(t) \in \Omega \ \forall t \) and \( \lim_{t \to \infty} x(t) = 0 \).

Now, consider the system \( \dot{x}(t) = Ax(t) + f_{23}(x(t)) \) where \( A \) is Hurwitz, and \( f_{23} \) is a polynomial with quadratic and cubic terms. For any quadratic, positive-definite \( l_1, l_2 \) and \( p \), the equations (1) are feasible using quadratic \( V \), constant \( s_1, s_3 \) and quadratic \( s_2 \) (suboptimal, feasible values are easily determined from \( A \) and \( f_{23} \)). Consequently, if the local stability of an equilibrium point of a system with a cubic vector field is decidable using linearized analysis, then the SOS region-of-attraction analysis will always yield a quantitative, certified, inner estimate of the region of attraction.

**SOS \( L_2 \) gain analysis:** For the driven system \( \dot{x} = f(x, w), z = h(x) \), if there exist positive-definite, radially unbounded function \( l_1 \), SOS polynomial \( s_1 \), a polynomial function \( V \), and
positive constants $\gamma$ and $R$ such that $V(0) = 0$ and
\begin{equation}
V - l_1 \in \Sigma[x],
\end{equation}
\begin{equation}
- \left[ (R^2 - V) s_1 + \nabla V f - w^T w + \frac{1}{\gamma} h^T h \right] \in \Sigma[x],
\end{equation}
then for all $w$, with $\|w\|_{2,T} \leq R$, the solution from $x(0) = 0$ satisfies $V(x(t)) \leq R^2$ and $\|z\|_{2,T} \leq \gamma \|w\|_{2,T}$.

Now consider the system $\dot{x} = Ax + f_2(x) + f_3(x) + [B + g_1(x)] w$ and $z = Cx + h_2(x)$ where $g_1$ is purely linear, $f_2$ and $h_2$ are purely quadratic, and $f_3$ is purely cubic.

Suppose the linearization $(A, B$ and $C)$ has $A$ Hurwitz, and $\|C(sI - A)^{-1}B\|_\infty < \gamma$. For any quadratic, positive-definite $l_1$, there exist $R > 0$ such that the equations (2) are feasible (possibly after scaling the state coordinates, $x \leftarrow \alpha x$, by a computable scalar $\alpha > 0$) using quadratic $V$, and quadratic $s_1$ (suboptimal, feasible values are easily determined from $A$, $f_2, f_3$, etc.). Consequently, if the local input/output gain around an equilibrium point of a system with a cubic vector field is bounded using linearized analysis, then the SOS $L_2$ gain analysis will always yield a quantitative, certified, ball of disturbances such that the same gain bound holds for the nonlinear system.

Acknowledgment/Disclaimer
This work was sponsored (in part) by the Air Force Office of Scientific Research, USAF, under grant/contract number FA9550-05-1-0266. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the Air Force Office of Scientific Research or the U.S. Government.

Personnel Supported During Duration of Grant
Weehong Tan (UC Berkeley, 04/2005-01/2006, externally funded, DSO National Laboratories, Singapore, but implicitly supported through the supervisory role of Packard, PhD in January 2006)

Ufuk Topcu (UC Berkeley, 08/2005-07/2008, PhD in July 2008)

Tim Wheeler (UC Berkeley, 08/2005-12/05, 5/06-8/06)

Publications


**Honors and Awards Received**

IEEE Fellow (Packard)

**AFRL Point of Contact**

Balas spoke at SAE *Aerospace, Control and Guidance Systems Committee* meeting, in Cocoa Beach FL, Oct 9-12, 2007. Air Force attendees included Dave Doman, Scott Wells and Marc Steinberg and many others from NASA and FAA.

Balas spoke at Honeywell Automation and Control Solutions 2007 Fellows Symposium in Minneapolis, MN on Oct 2, 2007. Honeywell fellows in the flight, home and building, and process control area were in attendance.

Packard, Balas and Seiler visited The Boeing Company in St. Louis MO on Dec 12, 2007 to give a presentation on the AFOSR V&V SOS research and to discuss possible collaborations between Boeing and UCB/UMN. Boeing attendees included Kevin Wise, Eugene Lavretsky, Jeff Ratliff and Steve Ramsey. Lt. Col. James McCormick, USAF, currently with DARPA TTO also attended part of the presentation.

January 2008 presentation at NASA Langley (Packard, Balas)

May 2008 presentation at NASA Ames (Packard)

**Transitions**

None yet.

**New Discoveries**

Nonaffine parameter uncertainty in region-of-attraction and gain analysis. Special minimum-volume polytopic coverings of graphs of polynomial functions. Local small-gain theorems, and application to region-of-attraction analysis with unmodeled dynamics. Superior performance on all examples tested thusfar.