10 Distributions

In this section we learn some tricks that allow us to redo the calculation in the previous section for general systems (not just second order). Distributions are mathematical constructions that, in a rigorous manner, allow us to repeat the rigorous calculation we just did in section 9 in a less tedious fashion.

10.1 Introduction

Recall our goal: determine the response of

\[ y^{[n]}(t) + a_1y^{[n-1]}(t) + a_2y^{[n-2]}(t) + \cdots + a_ny(t) = \]
\[ b_0u^{[n]}(t) + b_1u^{[n-1]}(t) + b_2u^{[n-2]}(t) + \cdots + b_nu(t) \]  \hspace{1cm} (55)

subject to \( u(t) = \mu(t) \) (a unit step at \( t = 0 \)) and initial conditions \( y(0^-) = y_0^0, y^{[1]}(0^-) = y_0^1, y^{[2]}(0^-) = y_0^2, \ldots, y^{[n-1]}(0^-) = y_0^{n-1} \). Here \( 0^- \) refers to the time just before the unit-step input is applied. So, the system is placed in initial conditions, and released, the release time being denoted \( 0^- \). At that instant, the input’s value is 0, so \( u(0^-) = 0 \). An infinitesimal time later, at \( 0^+ \), the input value is changed to +1. The input is actually at unit-step at \( t = 0 \), and the initial conditions are known just before the step-input is applied.

The difficulty is that the right-hand side of (55) is not classically defined, since \( u \) is not differentiable. However, we may obtain the solution by considering a sequence of problems, with smooth approximations to the unit step, and obtain the solution as a limiting process. Such a procedure goes as follows:

1. Define a \( n \)-times differentiable family of functions \( \mu_\epsilon \) that have the property

\[ \mu_\epsilon(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > \epsilon \end{cases} \]

and which converge to the unit step \( \mu \) as \( \epsilon \to 0 \).

2. Compute solution \( y_\epsilon(t) \) to the differential equation

\[ y^{[n]}(t) + a_1y^{[n-1]}(t) + a_2y^{[n-2]}(t) + \cdots + a_ny(t) = \]
\[ b_0u^{[n]}(t) + b_1u^{[n-1]}(t) + b_2u^{[n-2]}(t) + \cdots + b_nu(t) \]

subject to the forcing function \( u(t) = \mu_\epsilon(t) \) and initial conditions \( y(0) = y_0^0, y^{[1]}(0) = y_0^1, y^{[2]}(0) = y_0^2, \ldots, y^{[n-1]}(0) = y_0^{n-1} \).

3. Look at the values of \( y_\epsilon(\epsilon), y_\epsilon(\epsilon), \tilde{y}_\epsilon(\epsilon), \ldots, y_\epsilon^{[n-1]}(\epsilon) \). Take the limit as \( \epsilon \to 0 \), and get a relationship between the values of \( y, \dot{y}, \ddot{y}, \ldots, y^{[n-1]} \) at \( t = 0^- \) and \( t = 0^+ \).
4. For $t > 0$, the right hand side of the ODE in (55) is well defined (in fact, it’s a constant, $b_n$), and the solution can be determined from finding any particular solution, and combining with the family of homogeneous solutions to correctly match the conditions at $t = 0^+$. 

This procedure is tedious, though the following can be proven:

1. It gives the correct answer for the solution of the ODE subject to the unit step input

2. The final answer for the limits $y(t^k)(t)$ as $t \to 0$ are the same regardless of the form of $\mu$, as long as it satisfies the properties given.

Moreover, you can accomplish this in an easier fashion by using generalized functions, called distributions.

The most common distribution that is not a normal function is the Dirac-$\delta$ function. To gain intuition about this, consider an $\epsilon$-approximation to the unit-step function of the form

$$
\mu_{\epsilon}(t) = \begin{cases} 
0 & \text{for } t < 0 \\
\frac{1}{\epsilon} t & \text{for } 0 < t < \epsilon \\
1 & \text{for } t \geq 1
\end{cases}
$$

The derivative of this is

$$
\frac{d}{dt} \mu_{\epsilon}(t) = \begin{cases} 
0 & \text{for } t < 0 \\
\frac{1}{\epsilon} & \text{for } 0 < t < \epsilon \\
0 & \text{for } t \geq 1
\end{cases}
$$

Call this function $\delta_{\epsilon}$. Note that for all values of $\epsilon > 0$,

$$
\int_{-\infty}^{\infty} \delta_{\epsilon}(t) dt = 1
$$

and that for $t < 0$ and $t > \epsilon$, $\delta_{\epsilon}(t) = 0$. Moreover, for any continuous function $f$

$$
\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} f(t) \delta_{\epsilon}(t) dt = f(0)
$$

and for any $t \neq 0$, $\lim_{\epsilon \to 0} \delta_{\epsilon}(t) = 0$. Hence, in the limit we can imagine a “function” $\delta$ whose value at nonzero $t$ is 0, whose value at $t = 0$ is undefined, but whose integral is finite, namely 1.

Now, we can start another level smoother. Let $\delta_{\epsilon}$ be defined as

$$
\delta_{\epsilon}(t) := \begin{cases} 
0 & \text{for } t < 0 \\
\frac{t}{2\epsilon^2} & \text{for } 0 < t < \epsilon \\
\frac{\epsilon - t}{\epsilon^2} & \text{for } \epsilon < t < 2\epsilon \\
0 & \text{for } 2\epsilon < t
\end{cases}
$$
Note that independent of \( \varepsilon \) we have
\[
\int_{-\infty}^{\infty} \delta_\varepsilon(t)\,dt = 1
\]
and for any \( t \neq 0 \), \( \lim_{\varepsilon \to 0} \delta_\varepsilon(t) = 0 \). The derivative is well defined, and satisfies
\[
\frac{d\delta_\varepsilon}{dt} = \begin{cases} 
0 & \text{for } t < 0 \\
\frac{1}{\varepsilon} & \text{for } 0 < t < \varepsilon \\
-\frac{1}{\varepsilon} & \text{for } \varepsilon < t < 2\varepsilon \\
0 & \text{for } t > 2\varepsilon 
\end{cases}
\]

Note that for any continuous function \( f \), we have
\[
\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} f(t)\delta_\varepsilon(t)\,dt = f(0)
\]
and for any differentiable function \( f \), we have
\[
\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} f(t)\frac{d\delta_\varepsilon}{dt}\,dt = -\dot{f}(0)
\]
and
\[
\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \left| \frac{d\delta_\varepsilon}{dt} \right|\,dt = \infty
\]
In the limit, we get an even more singular function, \( \dot{\delta} \) (or \( \delta^{[1]} \)), which has the property that it is zero for all nonzero \( t \), not well defined for \( t = 0 \), and even the integral of the absolute value is not well defined.

If we start with a smoother (quadratics) version of \( \delta_\varepsilon \), then we can differentiate twice, getting \( \ddot{\delta}_\varepsilon \) and \( \ddot{\delta}_\varepsilon \), and look at their limiting behavior. Starting still smoother, we can continue the process, generating a collection of singular functions, \( \delta^{[k]} \), defined formally as
\[
\delta^{[0]}(t) := \delta(t) := \frac{d\mu}{dt}, \quad \delta^{[k]}(t) = \frac{d\delta^{[k-1]}}{dt}
\]
As we have seen in the first few, each \( \delta^{[k]} \) is “more singular” than the preceding one, \( \delta^{[k-1]} \). Hence, there is no manner in which several lower order derivatives of \( \delta \)-functions can be linearly combined to represent a \( \delta \)-function of higher order. In other words, the derivatives of \( \delta \) functions are a linearly independent set.

### 10.2 Procedure to get step response

Consider our general system differential equation (55), and suppose that \( u(t) = \mu(t) \). Then, the highest order singularity in the right-hand side is associated with the \( u^{[n]} \) term, which has \( \delta^{[n-1]} \). Hence, this is the highest order singularity which must occur on the left-hand side. If this order singularity occurs in any of the terms \( y, \dot{y}, \ldots, y^{[n-1]} \), then additional
differentiation will yield a higher order singularity in the \( y^{[n]} \) term, which is not possible. Hence, the only term on the left-hand side with \( \delta^{[n-1]} \) is \( y^{[n]} \). This suggests that \( y^{[n]} \) is of the form

\[
y^{[n]} = e_1 \delta^{[n-1]} + e_2 \delta^{[n-2]} + \cdots + e_n \delta + e_{n+1} \mu + f_n
\]

where \( f_n \) is a continuous function, and the constants \( e_1, e_2, \ldots, e_{n+1} \) need to be determined. Note that if this is the form of the \( n \)'th derivative of \( y \), then there are constraints on the lower order derivatives of \( y \) as well. Integrating, we get

\[
\begin{align*}
y^{[n]} &= e_1 \delta^{[n-1]} + e_2 \delta^{[n-2]} + \cdots + e_n \delta + e_{n+1} \mu + f_n \\
y^{[n-1]} &= e_1 \delta^{[n-2]} + e_2 \delta^{[n-2]} + \cdots + e_n \delta_n + e_{n+1} \mu + f_{n-1} \\
\vdots \\
y^{[1]} &= e_1 \delta + e_2 \mu + f_2 \\
y^{[0]} &= e_1 \mu + f_1
\end{align*}
\]

where each of the \( f_i \) are continuous functions. Plugging into the ODE, and equating the different singularities gives \( n \) equations in \( n \) unknowns, expressed in matrix form below

\[
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
a_1 & 1 & 0 & \cdots & 0 \\
a_2 & a_1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n-1} & a_{n-2} & \cdots & a_1 & 1
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2 \\
e_3 \\
\vdots \\
e_n
\end{bmatrix}
= \begin{bmatrix}
b_0 \\
b_1 \\
b_2 \\
\vdots \\
b_{n-1}
\end{bmatrix}
\tag{56}
\]

The matrix is always invertible, and the system of equations can be solved, yielding the \( e \) vector. It is easy to see that it can be solved recursively, from \( e_1 \) through \( e_n \) (and even \( e_{n+1} \)) as

\[
\begin{align*}
e_1 &= b_0 \\
e_2 &= b_1 - a_1 e_1 \\
e_3 &= b_2 - a_1 e_2 - a_2 e_1 \\
\vdots \\
e_k &= b_{k-1} - a_1 e_{k-1} - a_2 e_{k-2} - \cdots - a_{k-1} e_1
\end{align*}
\]

The \( e \) vector gives the discontinuity in each derivative of \( y \) at \( t = 0 \), namely

\[
\begin{align*}
y(0^+) &= y(0^-) + e_1 \\
\dot{y}(0^+) &= \dot{y}(0^-) + e_2 \\
y^{[2]}(0^+) &= y^{[2]}(0^-) + e_3 \\
\vdots \\
y^{[k]}(0^+) &= y^{[k]}(0^-) + e_{k+1} \\
\vdots \\
y^{[n-1]}(0^+) &= y^{[n-1]}(0^-) + e_n
\end{align*}
\]

Given these “new” initial conditions at \( t = 0^+ \), we can combine the family of all homogeneous solutions, with one particular solution (for instance, \( y_P(t) \equiv \frac{b_n}{a_n} \)) to match the initial
conditions at \( t = 0^+ \), completing our solution. You can/should write a general MatLab function M-file to compute all of these quantities for given row vectors

\[
A = \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} & a_n \end{bmatrix}, \quad B = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} & b_n \end{bmatrix}
\]

### 10.3 Problems

1. Consider

\[
y^{[3]}(t) + 3.8\ddot{y}(t) + 6.8\dot{y}(t) + 4y(t) = b_2\dddot{u}(t) + 3u(t)
\]

subject to the forcing function \( u(t) := \mu(t) \), and initial conditions \( y(0^-) = 0, \dot{y}(0^-) = 0, \ddot{y}(0^-) = 0 \). Follow the same instructions as in problem 1 on page 82, handling the 7 cases

\[
b_2 = -6, -3, -0.3, -0.03, 0.03, 0.3, 3, 6.
\]

Since this is a higher order problem, you will need to also determine \( \dddot{y}(0^+) \). \textbf{Hint:} One of the roots of the characteristic equation is \(-1\). Also, if you proceed symbolically, you end up with the coefficients of the homogeneous components being of the form

\[
c = M^{-1}v(b_2)
\]

where \( M \) is a \( 3 \times 3 \) matrix made up of the three roots of the characteristic polynomial, and \( v \) is a \( 3 \times 1 \) vector that depends on \( b_2 \). On paper, leave it as that (don’t bother computing the inverse). Then, for each of the 6 cases, plug in a particular value for \( b_2 \), and let MatLab compute the coefficients automatically. Set up your plotting script file to accept a \( 3 \times 1 \) vector of homogeneous coefficients. In your solutions, include any useful M-files that you write.

2. Consider the three differential equations

\[
\begin{align*}
y^{[4]}(t) + 5.8y^{[3]}(t) + 14.4\dddot{y}(t) + 17.6\ddot{y}(t) + 8y(t) &= u(t) \\
y^{[4]}(t) + 5.8y^{[3]}(t) + 14.4\dddot{y}(t) + 17.6\ddot{y}(t) + 8y(t) &= 2\dddot{u}(t) + 2\ddot{u}(t) + u(t) \\
y^{[4]}(t) + 5.8y^{[3]}(t) + 14.4\dddot{y}(t) + 17.6\ddot{y}(t) + 8y(t) &= 2\dddot{u}(t) - 2\ddot{u}(t) + u(t)
\end{align*}
\]

Suppose that each is subject to the forcing function \( u(t) := \mu(t) \), and initial conditions \( y(0^-) = 0, \dot{y}(0^-) = 0, \ddot{y}(0^-) = 0, y^{[3]}(0^-) = 0 \). Compute the roots (hint: one at \(-1\), one at \(-2\)), get final value of \( y(t) \), compute “new” conditions of \( y \) (and derivatives) at \( 0^+ \), and sketch solutions. Then, derive the exact expression for the solutions, and plot using MatLab.