12 Transfer functions

Associated with the linear system (input \( u \), output \( y \)) governed by the ODE

\[
y^{[n]}(t) + a_1 y^{[n-1]}(t) + \cdots + a_{n-1} y^{[1]}(t) + a_n y(t) = b_0 u^{[n]}(t) + b_1 u^{[n-1]}(t) + \cdots + b_{n-1} u^{[1]}(t) + b_n u(t)
\]

we write “in transfer function form”

\[
Y = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n U}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}
\]

The expression in (61) is interpreted to be equivalent to the ODE in (60), just a different way of writing the coefficients. The notation in (61) is suggestive of multiplication, and we will see that such an interpretation is indeed useful. The function

\[
G(s) := \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}
\]

is called the transfer function from \( u \) to \( y \), and is sometimes denoted \( G_{u \rightarrow y}(s) \) to indicate this. At this point, the expression in equation (61),

\[
Y = G_{u \rightarrow y}(s)U
\]

is nothing more than a new notation for the differential equation in (60). The differential equation has a well-defined meaning, and we understand what each term represents, and the meaning of the equality sign, \( = \). In the transfer function expression, (61), there is no specific meaning to the individual terms, or the equality symbol. The expression, as a whole, simply means the differential equation it is associated with.

In this section, we will see that, in fact, we can assign proper equality, and make algebraic substitutions and manipulations of transfer function expressions, which will aid our manipulation of linear differential equations. But all of that requires proof, and that is the purpose of this section.

12.1 Linear Differential Operators (LDOs)

Note that in the expression (61), the symbol \( s \) plays the role of \( \frac{d}{dt} \), and higher powers of \( s \) mean higher order derivatives, i.e., \( s^k \) means \( \frac{d^k}{dt^k} \). If \( z \) is a function of time, let the notation

\[
\left[ b_0 \frac{d^n}{dt^n} + b_1 \frac{d^{n-1}}{dt^{n-1}} + \cdots + b_{n-1} \frac{d}{dt} + b_n \right](z) := b_0 \frac{d^n}{dt^n} z + b_1 \frac{d^{n-1}}{dt^{n-1}} z + \cdots + b_{n-1} \frac{d}{dt} z + b_n z
\]

We will call this type of operation a linear differential operation, or LDO. For the purposes of this section, we will denote these by capital letters, say

\[
L := \left[ \frac{d^n}{dt^n} + a_1 \frac{d^{n-1}}{dt^{n-1}} + \cdots + a_{n-1} \frac{d}{dt} + a_n \right]
\]

\[
R := \left[ b_0 \frac{d^n}{dt^n} + b_1 \frac{d^{n-1}}{dt^{n-1}} + \cdots + b_{n-1} \frac{d}{dt} + b_n \right]
\]
Using this shorthand notation, we can write the original ODE in (64) as

\[ L(y) = R(u) \]

With each LDO, we naturally associate a polynomial. Specifically, if

\[ L := \left[ \frac{d^n}{dt^n} + a_1 \frac{d^{n-1}}{dt^{n-1}} + \cdots + a_{n-1} \frac{d}{dt} + a_n \right] \]

then \( p_L(s) \) is defined as

\[ p_L(s) := s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n \]

Similarly, with each polynomial, we associate an LDO – if

\[ q(s) := s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m \]

then \( L_q \) is defined as

\[ L_q := \left[ \frac{d^m}{dt^m} + b_1 \frac{d^{m-1}}{dt^{m-1}} + \cdots + b_{m-1} \frac{d}{dt} + b_m \right] \]

Therefore, if a linear system is governed by an ODE of the form \( L(y) = R(u) \), then the transfer function description is simply

\[ Y = \frac{p_R(s)}{p_L(s)} U \]

Similarly, if the transfer function description of a system is

\[ V = \frac{n(s)}{d(s)} W \]

then the ODE description is \( L_d(v) = L_n(w) \).

### 12.2 Algebra of Linear differential operations

Note that two successive linear differential operations can be done in either order. For example let

\[ L_1 := \left[ \frac{d^2}{dt^2} + 5 \frac{d}{dt} + 6 \right] \]

and

\[ L_2 := \left[ \frac{d^3}{dt^3} - 2 \frac{d^2}{dt^2} + 3 \frac{d}{dt} - 4 \right] \]
Then, on a differentiable signal $z$, simple calculations gives

$$L_1 (L_2(z)) = \left[ \frac{d^2}{dt^2} + 5 \frac{d}{dt} + 6 \right] \left( \left[ \frac{d^3}{dt^3} - 2 \frac{d^2}{dt^2} + 3 \frac{d}{dt} - 4 \right] (z) \right)$$

$$= \left[ \frac{d^3}{dt^3} - 2 \frac{d^2}{dt^2} + 3 \frac{d}{dt} - 4 \right] \left( z^3 - 2 \dot{z} + 3 \ddot{z} - 4 z \right)$$

$$= \dot{z}^5 - 2 \dot{z}^4 + z^3 - 8 \dot{z}^2 + 15 z^2 - 20 z - 24 z$$

which is the same as

$$L_2 (L_1(z)) = \left[ \frac{d^3}{dt^3} - 2 \frac{d^2}{dt^2} + 3 \frac{d}{dt} - 4 \right] \left( \left[ \frac{d^3}{dt^3} - 2 \frac{d^2}{dt^2} + 3 \frac{d}{dt} - 4 \right] (z) \right)$$

$$= \dot{z}^5 + 3 \dot{z}^4 - z^3 - 2 \dot{z} - 2 \ddot{z} - 24 z$$

This equality is easily associated with the fact that multiplication of polynomials is a commutative operation, specifically

$$(s^2 + 5s + 6) (s^3 - 2s^2 + 3s - 4) = (s^3 - 2s^2 + 3s - 4) (s^2 + 5s + 6)$$

$$= s^5 + 3s^4 - s^3 - s^2 - 2s + 24$$

We will use the notation $[L_1 \circ L_2]$ to denote this composition of LDOs. The linear differential operator $L_1 \circ L_2$ is defined as operating on an arbitrary signal $z$ by

$$[L_1 \circ L_2](z) := L_1 (L_2(z))$$

Similarly, if $L_1$ and $L_2$ are LDOs, then the sum $L_1 + L_2$ is an LDO defined by its operation on a signal $z$ as $[L_1 + L_2](z) := L_1(z) + L_2(z)$.

It is clear that the following manipulations are always true for every differentiable signal $z$,

$$L (z_1 + z_2) = L (z_1) + L (z_2)$$

and

$$[L_1 \circ L_2](z) = [L_2 \circ L_1](z)$$

In terms of LDOs and their associated polynomials, we have the relationships

$$p_{L_1 + L_2}(s) = p_{L_1}(s) + p_{L_2}(s)$$

$$p_{[L_1 \circ L_2]}(s) = p_{L_1}(s)p_{L_2}(s)$$

In the next several subsections, we derive the LDO representation of an interconnection from the LDO representation of the subsystems.
12.3 Feedback Connection

The most important interconnection we know of is the basic feedback loop. It is also the easiest interconnection for which we derive the differential equation governing the interconnection from the differential equation governing the components.

Consider the simple unity-feedback system shown below

\[ r \xrightarrow{+} u \xrightarrow{S} y \]

Assume that system \( S \) is described by the LDO \( L(y) = D(u) \). The feedback interconnection yields \( u(t) = r(t) - y(t) \). Eliminate \( u \) by substitution, yielding an LDO relationship between \( r \) and \( y \)

\[ L(y) = D(r - y) = D(r) - D(y) \]

This is rearranged to the closed-loop LDO

\[ (L + D)(y) = D(r). \]

That’s a pretty simple derivation. Based on the ODE description of the closed-loop, we can immediately write the closed-loop transfer function,

\[ Y = \frac{p_D(s)}{p(L+D)(s)} R = \frac{p_D(s)}{p_D(s)} \frac{R}{p_L(s) + p_D(s)}. \]

Additional manipulation leads to further interpretation. Let \( G(s) \) denote the transfer function of \( S \), so \( G = \frac{p_D(s)}{p_L(s)} \). Then

\[ Y = \frac{p_D(s)}{p_L(s) + p_D(s)} R \]

\[ = \frac{p_D(s)}{1 + \frac{p_D(s)}{p_L(s)}} R \]

\[ = \frac{G(s)}{1 + G(s)} R \]

This can be interpreted rather easily. Based on the original system interconnection, redraw, replacing signals with their capital letter equivalents, and replacing the system \( S \) with its transfer function \( G \). This is shown below.
The diagram on the right is interpreted as a diagram of the equations \( U = R - Y \), and \( Y = GU \). Note that manipulating these as though they are arithmetic expressions gives

\[
Y = G(R - Y) \quad \text{after substituting for } U \\
(1 + G)Y = GR \quad \text{moving } GY \text{ over to left-hand side} \\
Y = \frac{G}{1 + G}R \quad \text{solving for } Y.
\]

This is is precisely what we want!

### 12.4 Cascade Connection

Suppose that we have two linear systems, as shown below,

\[
\begin{align*}
  &u \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
Hence, the governing equation for system $S_1$ is $L_1(y) = R_1(u)$, while the governing equation for system $S_2$ is $L_2(v) = R_2(y)$. Moreover, in terms of transfer functions, we have

$$G_1(s) = \frac{p_{R_1}(s)}{p_{L_1}(s)}, \quad G_2(s) = \frac{p_{R_2}(s)}{p_{L_2}(s)}$$

Now, apply the differential operation $R_2$ to the first system, leaving

$$R_2(L_1(y)) = R_2(R_1(u))$$

Apply the differential operation $L_1$ to system 2, leaving

$$L_1(L_2(v)) = L_1(R_2(y))$$

But, in the last section, we saw that two linear differential operations can be applied in any order, hence $L_1(R_2(y)) = R_2(L_1(y))$. This means that the governing differential equation for the cascaded system is

$$L_1(L_2(v)) = R_2(R_1(u))$$

which can be rearranged into

$$L_2(L_1(v)) = R_2(R_1(u))$$

or, in different notation

$$[L_2 \circ L_1](v) = [R_2 \circ R_1](u)$$

In transfer function form, this means

$$V = \frac{p_{[R_2 \circ R_1]}(s)}{p_{[L_2 \circ L_1]}(s)} U$$

$$= \frac{p_{R_2}(s)p_{R_1}(s)}{p_{L_2}(s)p_{L_1}(s)} U$$

$$= G_2(s)G_1(s)U$$

Again, this has a nice interpretation. Redraw the interconnection, replacing the signals with the capital letter equivalents, and the systems by their transfer functions.

The diagram on the right depicts the equations $Y = G_1U$, and $V = G_2Y$. Treating these as arithmetic equalities allows substitution for $Y$, which yields $V = G_2G_1U$, as desired.
Example: Suppose $S_1$ is governed by
\[ \ddot{y}(t) + 3\dot{y}(t) + y(t) = 3\dot{u}(t) - u(t) \]
and $S_2$ is governed by
\[ \ddot{v}(t) - 6\dot{v}(t) + 2v(t) = \dot{y}(t) + 4y(t) \]
Then for $S_1$ we have
\[ L_1 = \left[ \frac{d^2}{dt^2} + 3\frac{d}{dt} + 1 \right], \quad R_1 = \left[ 3\frac{d}{dt} - 1 \right], \quad G_1(s) = \frac{3s - 1}{s^2 + 3s + 1} \]
while for $S_2$ we have
\[ L_2 = \left[ \frac{d^2}{dt^2} - 6\frac{d}{dt} + 2 \right], \quad R_2 = \left[ \frac{d}{dt} + 4 \right], \quad G_2(s) = \frac{s + 4}{s^2 - 6s + 2} \]
The product of the transfer functions is easily calculated as
\[ G(s) := G_2(s)G_1(s) = \frac{3s^2 + 11s - 4}{s^4 - 3s^3 - 15s^2 + 2} \]
so that the differential equation governing $u$ and $v$ is
\[ v^{[4]}(t) - 3v^{[3]}(t) - 15v^{[2]}(t) + 2v(t) = 3u^{[2]}(t) + 11u^{[1]}(t) - 4u(t) \]
which can also be verified again, by direct manipulation of the ODEs.

12.5 Parallel Connection

Suppose that we have two linear systems, as shown below,

```
  u
   \|                  \|  y
S_1  \|                \|  +
 y_1  \|                \|  y
+ \|                \|  +
S_2  \|                \|  
 y_2
```

System $S_1$ is governed by
\[ y_1^{[n]}(t) + a_1y_1^{[n-1]}(t) + \cdots + a_ny_1(t) = b_0u^{[n]}(t) + b_1u^{[n-1]}(t) + \cdots + b_nu(t) \]
and denoted as $L_1(y_1) = R_1(u)$. Likewise, system $S_2$ is governed by
\[ y_2^{[m]}(t) + c_1y_2^{[m-1]}(t) + \cdots + c_my_2(t) = d_0u^{[m]}(t) + d_1u^{[m-1]}(t) + \cdots + d_mu(t) \]
and denoted $L_2(y_2) = R_2(u)$. 
Apply the differential operation $L_2$ to the governing equation for $S_1$, yielding

$$L_2 (L_1(y_1)) = L_2 (R_1(u)) \quad (62)$$

Similarly, apply the differential operation $L_1$ to the governing equation for $S_2$, yielding and

$$L_1 (L_2(y_2)) = L_1 (R_2(u))$$

But the linear differential operations can be carried out in either order, hence we also have

$$L_2 (L_1(y_2)) = L_1 (R_2(u)) \quad (63)$$

Add the expressions in (62) and (63), to get

$$L_2 (L_1(y)) = L_2 (L_1(y_1 + y_2)) = L_2 (L_1(y_1)) + L_2 (L_1(y_2)) = L_2 (R_1(u)) + L_1 (R_2(u)) = [L_2 \circ R_1](u) + [L_1 \circ R_2](u) = [L_2 \circ R_1 + L_1 \circ R_2](u)$$

In transfer function form this is

$$Y = \frac{p|L_2\circ R_1 + L_1\circ R_2|(s)}{p|L_2\circ L_1|(s)} U$$

$$= \frac{p|L_2\circ R_1|(s) + p|L_1\circ R_2|(s)}{pL_2(s)pL_1(s)} U$$

$$= \frac{pL_2(s)pR_1(s) + pL_1(s)pR_2(s)}{pL_2(s)pL_1(s)} U$$

$$= \frac{pR_1(s)}{pL_1(s)} + \frac{pR_2(s)}{pL_2(s)} U$$

$$= \left[ G_1(s) + G_2(s) \right] U$$

So, the transfer function of the parallel connection is the sum of the individual transfer functions.

This is extremely important! The transfer function of an interconnection of systems is simply the algebraic gain of the closed-loop systems, treating individual subsystems as complex gains, with their “gain” taking on the value of the transfer function.

### 12.6 General Connection

The following steps are used for a general interconnection of systems, each governed by a linear differential equation relating the inputs and outputs.
- Redraw the block diagram of the interconnection. Change signals (lower-case) to upper case, and replace each system with its transfer function.
- Write down the equations, in transfer function form, that are implied by the diagram.
- Manipulate the equations as though they are arithmetic expressions. Addition and multiplication commute, and the distributive laws hold.

12.7 Systems with multiple inputs

Associated with the multi-input, single-output linear ODE

\[ L(y) = R_1(u) + R_2(w) + R_3(v) \]  

we write

\[ Y = \frac{p_{R_1}(s)}{p_L(s)} U + \frac{p_{R_2}(s)}{p_L(s)} W + \frac{p_{R_3}(s)}{p_L(s)} V \]

This may be manipulated algebraically

12.8 Problems

1. Find the transfer function from \( u \) to \( y \) for the systems governed by the differential equations

\( \dot{y}(t) = \frac{1}{T} [u(t) - y(t)] \)

\( \dot{y}(t) + a_1 y(t) = b_0 \dot{u}(t) + b_1 u(t) \)

\( \dot{y}(t) = u(t) \) (explain connection to Simulink icon for integrator...)

\( \ddot{y}(t) + 2 \zeta \omega_n \dot{y}(t) + \omega_n^2 y(t) = \omega_n^2 u(t) \)

2. (a) \( F, G, K \) and \( H \) are transfer functions of systems. A block diagram of an interconnection is shown below. The input \( r \) and output \( y \) are labeled with their corresponding capital letters. Find the transfer function from \( r \) to \( y \).

(b) For a single-loop feedback system, a rule for determining the closed-loop transfer function from an specific input to a specific output is

\[
\frac{\text{forward path gain}}{1 - \text{feedback path gain}}
\]

Explain how part 2a above is a “proof” of this fact.
3. (a) $F, G, K$ and $H$ are transfer functions of systems. A block diagram of an interconnection is shown below. The input $r$ and output $y$ are labeled with their corresponding capital letters. Find the transfer function from $r$ to $y$.

\[
\begin{array}{c}
R \\
F \downarrow \\
p \downarrow \\
G \downarrow \\
+ \downarrow \\
H \downarrow \\
Y \\
K
\end{array}
\]

(b) For a single-loop feedback system, with negative feedback, a rule for determining the closed-loop transfer function from an specific input to a specific output is

\[
\frac{\text{forward path gain}}{1 + \text{feedback path gain}}
\]

Explain how part 3a above is a “proof” of this fact.

4. $G$ and $K$ are transfer functions of systems. A block diagram of an interconnection is shown below. The input $r$ and output $y$ are labeled with their corresponding capital letters. Find the transfer function from $r$ to $y$, and express it in terms of $N_G, D_G, N_K, D_K$, the numerators and denominators of the transfer functions $G$ and $K$.

\[
\begin{array}{c}
R \\
F \downarrow \\
E \downarrow \\
G \downarrow \\
+ \downarrow \\
K \downarrow \\
Y
\end{array}
\]

5. A feedback connection of 4 systems is shown below. Let the capital letters also denote the transfer functions of each of the systems.

\[
\begin{array}{c}
r \\
E \downarrow \\
G \downarrow \\
F \downarrow \\
+ \downarrow \\
H \downarrow \\
y
\end{array}
\]

(a) Break this apart as shown below.

\[
\begin{array}{c}
r \\
E \downarrow \\
G \downarrow \\
+ \downarrow \\
\text{d} \downarrow \\
x
\end{array}
\]

What is the transfer function from $r$ and $d$ to $x$? Call your answers $G_1$ and $G_2$. 
(b) Now, draw the overall system as

\[ r \rightarrow G_1 \rightarrow + \rightarrow H \rightarrow + \rightarrow G_2 \rightarrow F \rightarrow y \]

In terms of \( G_1, G_2 \) and \( F \) and \( H \), what is the transfer function from \( r \) to \( y \)? Substitute for \( G_1 \) and \( G_2 \), and get the transfer function from \( r \) to \( y \) in terms of the original subsystems.

(c) In terms of numerators and denominators of the individual transfer functions (\( G(s) = \frac{N_G(s)}{D_G(s)} \), for example), what is the characteristic equation of the closed-loop system?

6. (a) Suppose that the transfer function of a controller, relating reference signal \( r \) and measurement \( y \) to control signal \( u \) is

\[ U = C(s) [R - Y] \]

Suppose that the plant has transfer function relating control signal \( u \) and disturbance \( d \) to output \( y \) as

\[ Y = G_3(s) [G_1(s)U + G_2(s)D] \]

Draw a simple diagram, and determine the closed-loop transfer functions relating \( r \) to \( y \) and \( d \) to \( y \).

(b) Carry out the calculations for

\[ C(s) = K_P + \frac{K_I}{s}, \quad G_1(s) = \frac{E}{\tau s + 1}, \quad G_2(s) = G, \quad G_3(s) = \frac{1}{ms + \alpha} \]

Directly from this closed-loop transfer function calculation, determine the differential equation for the closed-loop system, relating \( r \) and \( d \) to \( y \).

(c) Given the transfer functions for the plant and controller in (6b),

i. Determine the differential equation for the controller, which relates \( r \) and \( y \) to \( u \).

ii. Determine the differential equation for the plant, which relates \( d \) and \( u \) to \( y \).

iii. Combining these differential equations, eliminate \( u \) and determine the closed-loop differential equation relating \( r \) and \( d \) to \( y \).

7. Find the transfer function from \( e \) to \( u \) for the PI controller equations

\[
\begin{align*}
\dot{z}(t) &= e(t) \\
u(t) &= K_P e(t) + K_I \dot{z}(t)
\end{align*}
\]
8. Suppose that the transfer function of a controller, relating reference signal \( r \) and measurement \( y_m \) to control signal \( u \) is

\[ U = C(s) [R - Y_M] \]

Suppose that the plant has transfer function relating control signal \( u \) and disturbance \( d \) to output \( y \) as

\[ Y = [G_1(s)U + G_2(s)D] \]

Suppose the measurement \( y_m \) is related to the actual \( y \) with additional noise \( (n) \), and a filter (with transfer function \( F \))

\[ Y_M = F(s) [Y + N] \]

(a) Draw a block diagram

(b) In one calculation, determine the 3 closed-loop transfer functions relating inputs \( r, d \) and \( n \) to the output \( y \).

(c) In one calculation, determine the 3 closed-loop transfer functions relating inputs \( r, d \) and \( n \) to the control signal \( u \).

9. A first order system has a transfer function

\[ G(s) = \frac{\gamma}{\tau s + 1} \]

(a) What is the differential equation relating the input and output?

(b) Under what conditions is the system stable?

(c) If the system is stable, what is the time-constant of the system?

10. Recall that if systems are connected in parallel (same input, and outputs add together) then the transfer function of the parallel connection is the sum of the transfer functions.

Consider the three different, complicated transfer functions

\[
G_1(s) = \frac{0.05s^{4} + 0.394s^{3} + 7.868s^{2} + 14.43s + 64}{0.04s^{5} + 1.184s^{4} + 7.379s^{3} + 73.19s^{2} + 95.36s + 64}
\]

\[
G_2(s) = \frac{0.05s^{4} + 2.536s^{3} + 64.36s^{2} + 87.87s + 64}{0.04s^{5} + 1.184s^{4} + 7.379s^{3} + 73.19s^{2} + 95.36s + 64}
\]

\[
G_3(s) = \frac{0.9s^{4} + 4.27s^{3} + 65.97s^{2} + 88.42s + 64}{0.04s^{5} + 1.184s^{4} + 7.379s^{3} + 73.19s^{2} + 95.36s + 64}
\]

The step responses of all three systems are shown on the next page. The frequency responses (Magnitude and Phase) are also shown.

Although it may not be physically motivated, mathematically, each \( G_i \) can be decomposed additively as as (you do not need to verify this)

\[
G_1(s) = 0.9 \cdot \frac{1}{s^{2} + 1.4s + 1} + 0.05 \cdot \frac{64}{s^{2} + 3.2s + 64} + 0.05 \cdot \frac{1}{0.04s + 1}
\]
\[ G_2(s) = 0.05 \frac{1}{s^2 + 1.4s + 1} + 0.9 \frac{64}{s^2 + 3.2s + 64} + 0.05 \frac{1}{0.04s + 1} \]

\[ G_3(s) = 0.05 \frac{1}{s^2 + 1.4s + 1} + 0.05 \frac{64}{s^2 + 3.2s + 64} + 0.9 \frac{1}{0.04s + 1} \]

Based on this information, match up each \( G_i \) to its step response, frequency response magnitude, and frequency response phase.
11. Assume $G_1, G_2$ and $H$ are transfer functions of linear systems. Compute the transfer function from $R$ to $Y$ in the figure below.

12. Read about the command `tf` in Matlab. Use the HTML help (available from the menubar, under `Help`, as well as the command-line (via `>> help tf`).

13. Read about the command `step` in Matlab. Use the HTML help (available from the menubar, under `Help`, as well as the command-line (via `>> help step`).

14. Execute the following commands

```matlab
>> sys1 = tf(1,[1 5.8 14.4 17.6 8])
>> sys2 = tf([2 2 1],[1 5.8 14.4 17.6 8])
>> sys3 = tf([2 -2 1],[1 5.8 14.4 17.6 8])
>> step(sys1,sys2,sys3)
```

Relate these to problem 2 on page 78 which you recently did and turned in.
15. Execute the commands

```matlab
>> sys1.den
>> class(sys1.den)
>> sys1.den{1}
>> class(sys1.den{1})
>> size(sys1.den{1})
>> roots(sys1.den{1})
>> pole(sys1)
```

Explain what is being referenced and displayed. Recall that the “poles” of a transfer function are the roots of the associated characteristic polynomial.

16. Read about the command `frd` in Matlab. Use the HTML help (available from the menubar, under Help, as well as the command-line (via `>> help frd`).

```matlab
>> OMEGA = logspace(-2,2,200);
>> G1 = frd(sys1,OMEGA);
>> G2 = frd(sys2,OMEGA);
>> G3 = frd(sys3,OMEGA);
```

17. Read about the commands `bode` and `bodemag` in Matlab.

18. Execute

```matlab
>> figure(1); bode(G1,G2,G3);
>> figure(2); bode(sys1,sys2,sys3);
```

Explain what is being plotted.

19. Read about the command `feedback` in Matlab. Use the HTML help (available from the menubar, under Help, as well as the command-line (via `>> help feedback`).

20. **Scenario:** Suppose process model is an integrator. Design a PI controller so that the closed-loop roots have $\omega_n = 1, \xi = 0.707$. Suppose that, in reality, the process is not just an integrator, but also has a first-order response, with time constant $\tau$. Plot the effect on the closed-loop for $\tau$ between 0.01 and 0.5 ($\approx 10^{-0.3}$).

Execute the following commands, and explain, in detail, what is being computed and plotted and how it relates to the scenario described above.

```matlab
wndes = 1; xides = 0.707;
C = tf([2*xides*wndes wndes*wndes],[1 0]);
tauvec = logspace(-2,-.3,5);
for i=1:length(tauvec)
```
P = tf([1],[1 0])*tf([1],[tauvec(i) 1]);
clp = feedback(P*C,1);
figure(1)
step(clp,8);
hold on
clp = feedback(C,P);
figure(2)
step(clp,8);
hold on
clp = feedback(P,C);
figure(3)
step(clp,8);
hold on
figure(4)
bode(clp);
hold on
end
figure(1); hold off; figure(2); hold off;
figure(3); hold off; figure(4); hold off;

Explain, in detail, what is being computed and plotted.