19 Linear Systems and Time-Invariance

We have studied the governing equations for many types of systems. Usually, these are nonlinear differential equations which govern the evolution of the variables as time progresses. In section 18.2, we saw that we could “linearize” a nonlinear system about an equilibrium point, to obtain a linear differential equation which governs the approximate behavior of the system near the equilibrium point. Also, in some systems, the governing equations were already linear. Linear differential equations are important class of systems to study, and these are the topic of the next several section.

19.1 Linearity of solution

Consider a vector differential equation
\[
\dot{x}(t) = A(t)x(t) + B(t)d(t)
\]
\[
x(t_0) = x_0
\]
(78)

where for each \( t \), \( A(t) \in \mathbb{R}^{n \times n} \), \( B(t) \in \mathbb{R}^{n \times m} \), and for each time \( t \), \( x(t) \in \mathbb{R}^n \) and \( d(t) \in \mathbb{R}^m \).

Claim: The solution function \( x(\cdot) \), on any interval \( [t_0, t_1] \) is a linear function of the pair \( (x_0, d(\cdot)[t_0, t_1]) \).

The precise meaning of this statement is as follows: Pick any constants \( \alpha, \beta \in \mathbb{R} \),

- if \( x_1 \) is the solution to (81) starting from initial condition \( x_{1,0} \) (at \( t_0 \)) and forced with input \( d_1 \), and
- if \( x_2 \) is the solution to (81) starting from initial condition \( x_{2,0} \) (at \( t_0 \)) and forced with input \( d_2 \),

then \( \alpha x_1 + \beta x_2 \) is the solution to (81) starting from initial condition \( \alpha x_{1,0} + \beta x_{2,0} \) (at \( t_0 \)) and forced with input \( \alpha d_1(t) + \beta d_2(t) \).

This can be easily checked, note that for every time \( t \), we have
\[
\dot{x}_1(t) = A(t)x_1(t) + B(t)d_1(t)
\]
\[
\dot{x}_2(t) = A(t)x_2(t) + B(t)d_2(t)
\]

Multiply the first equation by the constant \( \alpha \) and the second equation by \( \beta \), and add them together, giving
\[
\frac{d}{dt}[\alpha x_1(t) + \beta x_2(t)] = \alpha \dot{x}_1(t) + \beta \dot{x}_2(t)
\]
\[
= \alpha [A x_1(t) + B d_1(t)] + \beta [A x_2(t) + B d_2(t)]
\]
\[
= A [\alpha x_1(t) + \beta x_2(t)] + B [\alpha d_1(t) + \beta d_2(t)]
\]
which shows that the linear combination $\alpha x_1 + \beta x_2$ does indeed solve the differential equation.

The initial condition is easily checked. Finally, the existence and uniqueness theorem for differential equations tells us that this $(\alpha x_1 + \beta x_2)$ is the only solution which satisfies both the differential equation and the initial conditions.

Linearity of the solution in the pair (initial condition, forcing function) is often called **The Principal of Superposition** and is an extremely useful property of linear systems.

### 19.2 Time-Invariance

A separate issue, unrelated to linearity, is time-invariance. A system described by $\dot{x}(t) = f(x(t), d(t), t)$ is called **time-invariant** if (roughly) the behavior of the system does depend *explicitly* on the absolute time. In other words, shifting the time axis does not affect solutions.

Precisely, suppose that $x_1$ is a solution for the system, starting from $x_{10}$ at $t = t_0$ subject to the forcing $d_1(t)$, defined for $t \geq t_0$.

Now, let $\tilde{x}$ be the solution to the equations starting from $x_0$ at $t = t_0 + \Delta$, subject to the forcing $d(t) := d(t - \Delta)$, defined for $t \geq t_0 + \Delta$. Suppose that for all choices of $t_0, \Delta, x_0$ and $d(\cdot)$, the two responses are related by $\tilde{x}(t) = x(t - \Delta)$ for all $t \geq t_0 + \Delta$. Then the system described by $\dot{x}(t) = f(x(t), d(t), t)$ is called **time-invariant**.

In practice, the easiest manner to recognize time-invariance is that the right-hand side of the state equations (the first-order differential equations governing the process) do not explicitly depend on time. For instance, the system

\[
\begin{align*}
\dot{x}_1(t) &= 2 * x_2(t) - \sin [x_1(t)x_2(t)d_2(t)] \\
\dot{x}_2(t) &= -|x_2(t)| - x_2(t)d_1(t)
\end{align*}
\]

is nonlinear, yet time-invariant.