21 Linear State-Space Representations

First, let’s describe the most general type of dynamic system that we will consider/encounter in this class. Systems may have many inputs, and many outputs. Specifically, \( m \) inputs (denoted by \( d \)) and \( q \) outputs (denoted by \( e \)), and the input/output behavior is governed by a set of \( n \), 1st order, coupled differential equations, of the form

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t) \\
\vdots \\
\dot{x}_n(t) \\
e_1(t) \\
e_2(t) \\
\vdots \\
e_q(t)
\end{bmatrix} =
\begin{bmatrix}
f_1 (x_1(t), x_2(t), \ldots, x_n(t), d_1(t), d_2(t), \ldots, d_m(t), t) \\
f_2 (x_1(t), x_2(t), \ldots, x_n(t), d_1(t), d_2(t), \ldots, d_m(t), t) \\
\vdots \\
f_n (x_1(t), x_2(t), \ldots, x_n(t), d_1(t), d_2(t), \ldots, d_m(t), t) \\
h_1 (x_1(t), x_2(t), \ldots, x_n(t), d_1(t), d_2(t), \ldots, d_m(t), t) \\
h_2 (x_1(t), x_2(t), \ldots, x_n(t), d_1(t), d_2(t), \ldots, d_m(t), t) \\
\vdots \\
h_q (x_1(t), x_2(t), \ldots, x_n(t), d_1(t), d_2(t), \ldots, d_m(t), t)
\end{bmatrix}
\]  

(79)

where the functions \( f_i, h_i \) are given functions of the \( n \) variables \( x_1, x_2, \ldots, x_n \), the \( m \) variables \( d_1, d_2, \ldots, d_m \) and also explicit functions of \( t \).

For shorthand, we write (79) as

\[
\begin{aligned}
\dot{x}(t) &= f (x(t), d(t), t) \\
e(t) &= h (x(t), d(t), t)
\end{aligned}
\]  

(80)

Given an initial condition vector \( x_0 \), and a forcing function \( d(t) \) for \( t \geq t_0 \), we wish to solve for the solutions

\[
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
\vdots \\
x_n(t)
\end{bmatrix},
\begin{bmatrix}
e_1(t) \\
e_2(t) \\
\vdots \\
e_q(t)
\end{bmatrix}
\]

on the interval \([t_0, t_F]\), given the initial condition

\[
x(t_0) = x_0.
\]

and the input forcing function \( d(\cdot) \).

If the functions \( f \) and \( u \) are “reasonably” well behaved in \( x \) and \( t \), then the solution exists, is continuous, and differentiable at all points.

21.1 Linearity of solution

Consider a vector differential equation

\[
\begin{aligned}
\dot{x}(t) &= A(t)x(t) + B(t)d(t) \\
x(t_0) &= x_0
\end{aligned}
\]  

(81)
where for each \( t \), \( A(t) \in \mathbb{R}^{n \times n} \), \( B(t) \in \mathbb{R}^{n \times m} \), and for each time \( t \), \( x(t) \in \mathbb{R}^n \) and \( d(t) \in \mathbb{R}^m \).

**Claim:** The solution function \( x(\cdot) \), on any interval \([t_0, t_1]\) is a **linear function of the pair** \((x_0, d(t)_{[t_0, t_1]})\).

The precise meaning of this statement is as follows: Pick any constants \( \alpha, \beta \in \mathbb{R} \),

- if \( x_1 \) is the solution to (81) starting from initial condition \( x_{1,0} \) (at \( t_0 \)) and forced with input \( d_1 \), and
- if \( x_2 \) is the solution to (81) starting from initial condition \( x_{2,0} \) (at \( t_0 \)) and forced with input \( d_2 \),

then \( \alpha x_1 + \beta x_2 \) is the solution to (81) starting from initial condition \( \alpha x_{1,0} + \beta x_{2,0} \) (at \( t_0 \)) and forced with input \( \alpha d_1(t) + \beta d_2(t) \).

This can be easily checked, note that for every time \( t \), we have

\[
\begin{align*}
x_1(t) &= A(t)x_1(t) + B(t)d_1(t) \\
x_2(t) &= A(t)x_2(t) + B(t)d_2(t)
\end{align*}
\]

Multiply the first equation by the constant \( \alpha \) and the second equation by \( \beta \), and add them together, giving

\[
\frac{d}{dt} [\alpha x_1(t) + \beta x_2(t)] = \alpha \dot{x}_1(t) + \beta \dot{x}_2(t)
= \alpha [A x_1(t) + B d_1(t)] + \beta [A x_2(t) + B d_2(t)]
= A [\alpha x_1(t) + \beta x_2(t)] + B [\alpha d_1(t) + \beta d_2(t)]
\]

which shows that the linear combination \( \alpha x_1 + \beta x_2 \) does indeed solve the differential equation.

The initial condition is easily checked. Finally, the existence and uniqueness theorem for differential equations tells us that this \((\alpha x_1 + \beta x_2)\) is the only solution which satisfies both the differential equation and the initial conditions.

Linearity of the solution in the pair (initial condition, forcing function) is often called **The Principal of Superposition** and is an extremely useful property of linear systems.

### 21.2 Time-Invariance

A separate issue, unrelated to linearity, is time-invariance. A system described by \( \dot{x}(t) = f(x(t), d(t), t) \) is called **time-invariant** if (roughly) the behavior of the system does depend explicitly on the absolute time. In other words, shifting the time axis does not affect solutions.

Precisely, suppose that \( x_1 \) is a solution for the system, starting from \( x_{10} \) at \( t = t_0 \) subject to the forcing \( d_1(t) \), defined for \( t \geq t_0 \).
Now, let $\tilde{x}$ be the solution to the equations starting from $x_0$ at $t = t_0 + \Delta$, subject to the forcing $d(t) := d(t - \Delta)$, defined for $t \geq t_0 + \Delta$. Suppose that for all choices of $t_0, \Delta, x_0$ and $d(\cdot)$, the two responses are related by $\tilde{x}(t) = x(t - \Delta)$ for all $t \geq t_0 + \Delta$. Then the system described by $\dot{x}(t) = f(x(t), d(t), t)$ is called time-invariant.

In practice, the easiest manner to recognize time-invariance is that the right-hand side of the state equations (the first-order differential equations governing the process) do not explicitly depend on time. For instance, the system

\[
\begin{align*}
\dot{x}_1(t) &= 2 \times x_2(t) - \sin[x_1(t)x_2(t) d_2(t)] \\
\dot{x}_2(t) &= -|x_2(t)| - x_2(t) d_1(t)
\end{align*}
\]

is nonlinear, yet time-invariant.

For the next few sections, we focus entirely on systems that are linear and time-invariant. This is not any different than the systems governed by our standard ODE model, introduced in Section 7. However, the representation is different, namely lots of coupled 1st order equations, rather than one big n’th order equation. We will derive all of the same ideas from this perspective: general solution, stability, frequency response, transfer function.
21.3 Matrix Exponential

Recall that for the scalar differential equation
\[ \dot{x}(t) = ax(t) + bu(t) \]
\[ x(t_0) = x_0 \]
the solution for \( t \geq t_0 \) is given by the formula
\[ x(t) = e^{a(t-t_0)}x_0 + \int_{t_0}^{t} e^{a(t-\tau)}bu(\tau)d\tau \]

What makes this work is the special structure of the exponential function, namely that
\[ \frac{d}{dt}e^{at} = ae^{at}. \]

Now consider a vector differential equation
\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ x(t_0) = x_0 \]
(82)
where \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \) are constant matrices, and for each time \( t \), \( x(t) \in \mathbb{R}^n \) and \( u(t) \in \mathbb{R}^m \). The solution can be derived by proceeding analogously to the scalar case.

For a matrix \( A \in \mathbb{C}^{n \times n} \) define a matrix function of time, \( e^{At} \in \mathbb{C}^{n \times n} \) as
\[ e^{At} := \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots \]

This is exactly the same as the definition in the scalar case. Now, for any \( T > 0 \), every element of this matrix power series converges absolutely and uniformly on the interval \([0, T]\). Hence, it can be differentiated term-by-term to correctly get the derivative. Therefore
\[ \frac{d}{dt}e^{At} := \sum_{k=0}^{\infty} \frac{k}{k!}t^{k-1}A^k = A + tA^2 + \frac{t^2}{2!}A^3 + \cdots = A \left( I + tA + \frac{t^2}{2!}A^2 + \cdots \right) = Ae^{At} \]

This is the most important property of the function \( e^{At} \). Also, in deriving this result, \( A \) could have been pulled out of the summation on either side. Summarizing these important identities,
\[ e^{A0} = I_n, \quad \frac{d}{dt}e^{At} = Ae^{At} = e^{At}A \]

So, the matrix exponential has properties similar to the scalar exponential function. However, there are two important facts to watch out for:
- **WARNING:** Let $a_{ij}$ denote the $(i,j)$th entry of $A$. The $(i,j)$th entry of $e^{At}$ IS NOT EQUAL TO $e^{a_{ij}t}$. This is most convincingly seen with a nontrivial example. Consider 

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

A few calculations show that 

$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad \ldots \quad A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

The definition for $e^{At}$ is 

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

Plugging in, and evaluating on a term-by-term basis gives 

$$e^{At} = \begin{bmatrix} 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots & 0 + t + 2\frac{t^2}{2!} + 3\frac{t^3}{3!} + \cdots \\ 0 + 0 + 0 + 0 + \cdots & 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \end{bmatrix}$$

The $(1,1)$ and $(2,2)$ entries are easily seen to be the power series for $e^t$, and the $(2,1)$ entry is clearly 0. After a few manipulations, the $(1,2)$ entry is $te^t$. Hence, in this case

$$e^{At} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

which is very different than the element-by-element exponentiation of $At$, 

$$\begin{bmatrix} e^{a_{11}t} & e^{a_{12}t} \\ e^{a_{21}t} & e^{a_{22}t} \end{bmatrix} = \begin{bmatrix} e^t & e^t \\ 1 & e^t \end{bmatrix}$$

- **WARNING:** In general, 

$$e^{(A_1+A_2)t} \neq e^{A_1t}e^{A_2t}$$

unless $t = 0$ (trivial) or $A_1A_2 = A_2A_1$. However, the identity 

$$e^{A(t_1+t_2)} = e^{At_1}e^{At_2}$$

is always true. Hence $e^{At}e^{-At} = e^{-At}e^{At} = I$ for all matrices $A$ and all $t$, and therefore for all $A$ and $t$, $e^{At}$ is invertible.
21.3.1 Diagonal $A$

If $A \in \mathbb{C}^{n \times n}$ is diagonal, then $e^{At}$ is easy to compute. Specifically, if $A$ is diagonal, it is easy to see that for any $k$,

$$A = \begin{bmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_n \end{bmatrix}, \quad A^k = \begin{bmatrix} \beta_1^k & 0 & \cdots & 0 \\ 0 & \beta_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_n^k \end{bmatrix}$$

In the power series definition for $e^{At}$, any off-diagonal terms are identically zero, and the $i$’th diagonal term is simply

$$[e^{At}]_{ii} = 1 + t\beta_i + \frac{t^2}{2!}\beta_i^2 + \frac{t^3}{3!}\beta_i^3 + \cdots = e^{\beta_i t}$$

21.3.2 Block Diagonal $A$

If $A_1 \in \mathbb{C}^{n_1 \times n_1}$ and $A_2 \in \mathbb{C}^{n_2 \times n_2}$, define

$$A := \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

**Question:** How is $e^{At}$ related to $e^{A_1 t}$ and $e^{A_2 t}$? Very simply – note that for any $k \geq 0$,

$$A^k = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}^k = \begin{bmatrix} A_1^k & 0 \\ 0 & A_2^k \end{bmatrix}$$

Hence, the power series definition for $e^{At}$ gives

$$e^{At} = \begin{bmatrix} e^{A_1 t} & 0 \\ 0 & e^{A_2 t} \end{bmatrix}$$

21.3.3 Effect of Similarity Transformations

For any invertible $T \in \mathbb{C}^{n \times n}$,

$$e^{T^{-1}AT} = T^{-1}e^{At}T$$

Equivalently,

$$Te^{T^{-1}AT}T^{-1} = e^{At}$$

This can easily be shown from the power series definition.

$$e^{T^{-1}AT} = I + t(T^{-1}AT) + \frac{t^2}{2!}(T^{-1}AT)^2 + \frac{t^3}{3!}(T^{-1}AT)^3 + \cdots$$
It is easy to verify that for every integer $k$
\[
(T^{-1}AT)^k = T^{-1}A^kT
\]
Hence, we have (with $I$ written as $T^{-1}IT$)
\[
e^{T^{-1}ATt} = T^{-1}IT + tT^{-1}AT + \frac{t^2}{2!}T^{-1}A^2T + \frac{t^3}{3!}T^{-1}A^3T + \cdots
\]
Pull $T^{-1}$ out on the left side, and $T$ on the right to give
\[
e^{T^{-1}ATt} = T^{-1}\left(I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots\right)T
\]
which is simply
\[
e^{T^{-1}ATt} = T^{-1}e^{At}T
\]
as desired.

21.3.4 Examples

Given $\bar{\beta} \in \mathbb{R}$, define
\[
A := \begin{bmatrix}
0 & \beta \\
-\beta & 0
\end{bmatrix}
\]
Calculate
\[
A^2 = \begin{bmatrix}
-\beta^2 & 0 \\
0 & -\beta^2
\end{bmatrix}
\]
and hence for any $k$,
\[
A^{2k} = \begin{bmatrix}
(-1)^k\beta^{2k} & 0 \\
0 & (-1)^k\beta^{2k}
\end{bmatrix}, \quad A^{2k+1} = \begin{bmatrix}
0 & (-1)^k\beta^{2k+1} \\
(-1)^{k+1}\beta^{2k+1} & 0
\end{bmatrix}
\]
Therefore, we can write out the first few terms of the power series for $e^{At}$ as
\[
e^{At} = \begin{bmatrix}
1 - \frac{1}{2}\beta^2t^2 + \frac{1}{4!}\beta^4t^4 - \cdots & \beta t - \frac{1}{3!}\beta^3t^3 + \frac{1}{5!}\beta^5t^5 - \cdots \\
-\beta t + \frac{1}{3!}\beta^3t^3 - \frac{1}{5!}\beta^5t^5 + \cdots & 1 - \frac{1}{2}\beta^2t^2 + \frac{1}{4!}\beta^4t^4 - \cdots
\end{bmatrix}
\]
which is recognized as
\[
e^{At} = \begin{bmatrix}
\cos \beta t & \sin \beta t \\
-\sin \beta t & \cos \beta t
\end{bmatrix}
\]
Similarly, suppose $\alpha, \beta \in \mathbb{R}$, and
\[
A := \begin{bmatrix}
\alpha & \beta \\
-\beta & \alpha
\end{bmatrix}
\]
Then, \( A \) can be decomposed as \( A = A_1 + A_2 \), where
\[
A_1 := \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \quad A_2 := \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}
\]

Note: in this special case, \( A_1 A_2 = A_2 A_1 \), hence
\[
e^{(A_1 + A_2)t} = e^{A_1t}e^{A_2t}
\]

Since \( A_1 \) is diagonal, we know \( e^{A_1t} \), and \( e^{A_2t} \) follows from our previous example. Hence
\[
e^{At} = \begin{bmatrix} e^{\alpha t} \cos \beta t & e^{\alpha t} \sin \beta t \\ -e^{\alpha t} \sin \beta t & e^{\alpha t} \cos \beta t \end{bmatrix}
\]

This is an important case to remember.

Finally, suppose \( \lambda \in \mathbb{F} \), and
\[
A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}
\]

We did a similar example in the first Warning of section 21.3. A few calculations show that
\[
A^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix}
\]

Hence, the power series gives
\[
e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}
\]

### 21.4 Problems

1. Determine \( e^{At} \) for the matrices
\[
A = \begin{bmatrix} -2.4 & 1.8 \\ -1.8 & -2.4 \end{bmatrix}
\]

Plot the solution to \( \dot{x}(t) = Ax(t) \) from the initial conditions
\[
x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}
\]

Plot this in 2 different manners: first, two plots (using subplot) of \( x_1(t) \) versus \( t \), and \( x_2(t) \) versus \( t \); and in another single plot of \( x_2 \) versus \( x_1 \). In each plot, there should be 4 lines. Use different linetypes (or colors) and make sure they are consistent across all of the figures.
21.4.1 Solution To State Equations

The matrix exponential is extremely important in the solution of the vector differential equation

\[ \dot{x}(t) = Ax(t) + Bu(t) \]  

starting from the initial condition \( x(t_0) = x_0 \). Now, consider the original equation in (83). We can derive the solution to the forced equation using the “integrating factor” method, proceeding in the same manner as in the scalar case, with extra care for the matrix-vector operations. Suppose a function \( x \) satisfies (83). Multiply both sides by \( e^{-At} \) to give

\[ e^{-At} \dot{x}(t) = e^{-At} Ax(t) + e^{-At} Bu(t) \]

Move one term to the left, leaving,

\[ e^{-At} Bu(t) = e^{-At} \dot{x}(t) - e^{-At} Ax(t) = e^{-At} \dot{x}(t) - Ae^{-At} x(t) = \frac{d}{dt} \left[ e^{-At} x(t) \right] \]

Since these two functions are equal at every time, we can integrate them over the interval \([t_0, t]\). Note that the right-hand side of (84) is an exact derivative, giving

\[ \int_{t_0}^{t_1} e^{-At} Bu(t) dt = \frac{d}{dt} \left[ e^{-At} x(t) \right] \bigg|_{t_0}^{t_1} \]

Note that \( x(t_0) = x_0 \). Also, multiply both sides by \( e^{At_1} \), to yield

\[ e^{At_1} \int_{t_0}^{t_1} e^{-At} Bu(t) dt = x(t_1) - e^{A(t_1-t_0)} x_0 \]

This is rearranged into

\[ x(t_1) = e^{A(t_1-t_0)} x_0 + \int_{t_0}^{t_1} e^{A(t_1-t)} Bu(t) dt \]

Finally, switch variable names, letting \( \tau \) be the variable of integration, and letting \( t \) be the right end point (as opposed to \( t_1 \)). In these new letters, the expression for the solution of the (83) for \( t \geq t_0 \), subject to initial condition \( x(t_0) = x_0 \) is

\[ x(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau \]

consisting of a free and forced response.
21.5 Eigenvalues, eigenvectors, stability

21.5.1 Diagonalization: Motivation

Recall two facts from Section 21.3: For diagonal matrices $\Lambda \in \mathbb{F}^{n \times n}$,

$$
\Lambda = \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{bmatrix} \Rightarrow e^{\Lambda t} = \begin{bmatrix}
e^{\lambda_1 t} & 0 & \cdots & 0 \\
0 & e^{\lambda_2 t} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & e^{\lambda_n t}
\end{bmatrix}
$$

and: If $A \in \mathbb{F}^{n \times n}$, and $T \in \mathbb{F}^{n \times n}$ is invertible, and $\tilde{A} := T^{-1}AT$, then

$$
e^{At} = Te^{\tilde{A}t}T^{-1}
$$

Clearly, for a general $A \in \mathbb{F}^{n \times n}$, we need to study the invertible transformations $T \in \mathbb{F}^{n \times n}$ such that $T^{-1}AT$ is a diagonal matrix.

Suppose that $T$ is invertible, and $\Lambda$ is a diagonal matrix, and $T^{-1}AT = \Lambda$. Moving the $T^{-1}$ to the other side of the equality gives $AT = T\Lambda$. Let $t_i$ denote the $i$'th column of the matrix $T$. Since $T$ is assumed to be invertible, none of the columns of $T$ can be identically zero, hence $t_i \neq \Theta_n$. Also, let $\lambda_i$ denote the $(i, i)$'th entry of $\Lambda$. The $i$'th column of the matrix equation $AT = T\Lambda$ is just

$$
At_i = t_i\lambda_i = \lambda_it_i
$$

This observation leads to the next section.

21.5.2 Eigenvalues

Definition: Given a matrix $A \in \mathbb{F}^{n \times n}$. A complex number $\lambda$ is an eigenvalue of $A$ if there is a nonzero vector $v \in \mathbb{C}^n$ such that

$$
Av = v\lambda = \lambda v
$$

The nonzero vector $v$ is called an eigenvector of $A$ associated with the eigenvalue $\lambda$.

Remark: Consider the differential equation $\dot{x}(t) = Ax(t)$, with initial condition $x(0) = v$. Then $x(t) = ve^{\Lambda t}$ is the solution (check that it satisfies initial condition and differential equation). So, an eigenvector is “direction” in the state-space such that if you start in the direction of the eigenvector, you stay in the direction of the eigenvector.

Note that if $\lambda$ is an eigenvalue of $A$, then there is a vector $v \in \mathbb{C}^n$, $v \neq \Theta_n$ such that

$$
Av = v\lambda = (\lambda I)v
$$

Hence

$$
(\lambda I - A)v = \Theta_n
$$
Since \( v \neq \Theta_n \), it must be that \( \det (\lambda I - A) = 0 \).

For an \( n \times n \) matrix \( A \), define a polynomial, \( p_A(\cdot) \), called the **characteristic polynomial** of \( A \) by

\[
p_A(s) := \det (\lambda I - A)
\]

Here, the symbol \( s \) is simply the indeterminate variable of the polynomial.

For example, take

\[
A = \begin{bmatrix}
  2 & 3 & -1 \\
-1 & -1 & -1 \\
  0 & 2 & 0
\end{bmatrix}
\]

Straightforward manipulation gives \( p_A(s) = s^3 - s^2 + 3s - 2 \). Hence, we have shown that the eigenvalues of \( A \) are necessarily roots of the equation

\[
p_A(s) = 0.
\]

For a general \( n \times n \) matrix \( A \), we will write

\[
p_A(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n
\]

where the \( a_1, a_2, \ldots, a_n \) are complicated products and sums involving the entries of \( A \). Since the characteristic polynomial of an \( n \times n \) matrix is a \( n \)th order polynomial, the equation \( p_A(s) = 0 \) has at most \( n \) distinct roots (some roots could be repeated). **Therefore, a \( n \times n \) matrix \( A \) has at most \( n \) distinct eigenvalues.**

Conversely, suppose that \( \lambda \in \mathbb{C} \) is a root of the polynomial equation

\[
p_A(s)|_{s=\lambda} = 0
\]

**Question:** Is \( \lambda \) an eigenvalue of \( A \)?

**Answer:** Yes. Since \( p_A(\lambda) = 0 \), it means that

\[
\det (\lambda I - A) = 0
\]

Hence, the matrix \( \lambda I - A \) is singular (not invertible). Therefore, by the matrix facts, the equation

\[
(\lambda I - A) v = \Theta_n
\]

has a **nonzero** solution vector \( v \) (which you can find by Gaussian elimination). This means that

\[
\lambda v = Av
\]

for a nonzero vector \( v \), which means that \( \lambda \) is an eigenvalue of \( A \), and \( v \) is an associated eigenvector.

We summarize these facts as:
A is a \( n \times n \) matrix

The characteristic polynomial of \( A \) is

\[
p_A(s) := \det (sI - A) = s^n + a_1s^{n-1} + \cdots + a_{n-1}s + a_n
\]

A complex number \( \lambda \) is an eigenvalue of \( A \) if and only if \( \lambda \) is a root of the “characteristic equation” \( p_a(\lambda) = 0 \).

Next, we have a useful fact from linear algebra: Suppose \( A \) is a given \( n \times n \) matrix, and \( (\lambda_1, v_1), (\lambda_2, v_2), \ldots, (\lambda_n, v_n) \) are eigenvalue/eigenvector pairs. So, for each \( i \), \( v_i \neq \Theta_n \) and \( Av_i = v_i\lambda_i \). **Fact:** If all of the \( \{\lambda_i\}_{i=1}^n \) are distinct, then the set of vectors

\[
\{v_1, v_2, \ldots, v_n\}
\]

are a linearly independent set. In other words, the matrix

\[
V := [v_1 \ v_2 \ \cdots \ v_n] \in \mathbb{C}^{n \times n}
\]

is invertible.

**Proof:** We’ll prove this for \( 3 \times 3 \) matrices – check your linear algebra book for the generalization, which is basically the same proof.

Suppose that there are scalars \( \alpha_1, \alpha_2, \alpha_3 \), such that

\[
\sum_{i=1}^{3} \alpha_i v_i = \Theta_3
\]

This means that

\[
\Theta_3 = (A - \lambda_3 I) \Theta_3 = (A - \lambda_3 I) \sum_{i=1}^{3} \alpha_i v_i = \alpha_1 (A - \lambda_3 I) v_1 + \alpha_2 (A - \lambda_3 I) v_2 + \alpha_3 (A - \lambda_3 I) v_3 = \alpha_1 (\lambda_1 - \lambda_3) v_1 + \alpha_2 (\lambda_2 - \lambda_3) v_2 + \Theta_3
\]  \(85\)

Now multiply by \( (A - \lambda_2 I) \), giving

\[
\Theta_3 = (A - \lambda_2 I) \Theta_3 = (A - \lambda_2 I) [\alpha_1 (\lambda_1 - \lambda_3) v_1 + \alpha_2 (\lambda_2 - \lambda_3) v_2] = \alpha_1 (\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2) v_1
\]

Since \( \lambda_1 \neq \lambda_3, \lambda_1 \neq \lambda_2, v_1 \neq \Theta_3 \), it must be that \( \alpha_1 = 0 \). Using equation \(85\), and the fact that \( \lambda_2 \neq \lambda_3, v_2 \neq \Theta_3 \) we get that \( \alpha_2 = 0 \). Finally, \( \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \Theta_3 \) (by assumption), and \( v_3 \neq \Theta_3 \), so it must be that \( \alpha_3 = 0 \). \( \# \)
21.5.3 Diagonalization Procedure

In this section, we summarize all of the previous ideas into a step-by-step diagonalization procedure for $n \times n$ matrices.

1. Calculate the characteristic polynomial of $A$, $p_A(s) := \det(sI - A)$.
2. Find the $n$ roots of the equation $p_A(s) = 0$, and call the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$.
3. For each $i$, find a nonzero vector $t_i \in \mathbb{C}_n$ such that
   $$(A - \lambda_i I) t_i = \Theta_n$$
4. Form the matrix
   $$T := [t_1 \ t_2 \ \cdots \ t_n] \in \mathbb{C}^{n \times n}$$
   (note that if all of the $\{\lambda_i\}_{i=1}^n$ are distinct from one another, then $T$ is guaranteed to be invertible).
5. Note that $AT = T\Lambda$, where
   $$\Lambda = \begin{bmatrix}
   \lambda_1 & 0 & \cdots & 0 \\
   0 & \lambda_2 & \cdots & 0 \\
   \vdots & \vdots & \ddots & \vdots \\
   0 & 0 & \cdots & \lambda_n
   \end{bmatrix}$$
6. If $T$ is invertible, then $T^{-1}AT = \Lambda$. Hence
   $$e^{At} = Te^{\Lambda t}T^{-1} = T \begin{bmatrix}
   e^{\lambda_1 t} & 0 & \cdots & 0 \\
   0 & e^{\lambda_2 t} & \cdots & 0 \\
   \vdots & \vdots & \ddots & \vdots \\
   0 & 0 & \cdots & e^{\lambda_n t}
   \end{bmatrix}T^{-1}$$

We will talk about the case of nondistinct eigenvalues later.

21.5.4 $e^{At}$ as $t \to \infty$

For the remainder of the section, assume $A$ has distinct eigenvalues.

- if all of the eigenvalues (which may be complex) of $A$ satisfy
  $$\text{Re}(\lambda_i) < 0$$
  then $e^{\lambda_i t} \to 0$ as $t \to \infty$, so all entries of $[e^{At}]$ decay to zero
• If there is one (or more) eigenvalues of $A$ with
  \[ \text{Re} (\lambda_i) \geq 0 \]
  then
  \[ e^{\lambda_i t} \rightarrow \text{bounded} \neq 0 \text{ as } t \rightarrow \infty \]
  Hence, some of the entries of $e^{At}$ either do not decay to zero, or, in fact, diverge to $\infty$.

So, the eigenvalues are an indicator (the key indicator) of stability of the differential equation
  \[ \dot{x}(t) = Ax(t) \]

• if all of the eigenvalues of $A$ have negative real parts, then from any initial condition $x_0$, the solution
  \[ x(t) = e^{At}x_0 \]
  decays to $\Theta_n$ as $t \rightarrow \infty$ (all coordinates of $x(t)$ decay to 0 as $t \rightarrow \infty$). In this case, $A$ is said to be a Hurwitz matrix.

• if any of the eigenvalues of $A$ have nonnegative real parts, then from some initial conditions $x_0$, the solution to
  \[ \dot{x}(t) = Ax(t) \]
  does not decay to zero.

### 21.5.5 Complex Eigenvalues

In many systems, the eigenvalues are complex, rather than real. This seems unusual, since the system itself (ie., the physical meaning of the state and input variables, the coefficients in the state equations, etc.) is very much Real. The procedure outlined in section 21.5.3 for matrix diagonalization is applicable to both real and complex eigenvalues, and if $A$ is real, all intermediate complex terms will cancel, resulting in $e^{At}$ being purely real, as expected. However, in the case of complex eigenvalues it may be more advantageous to use a different similarity transformation, which does not lead to a diagonalized matrix. Instead, it leads to a real block-diagonal matrix, whose structure is easily interpreted.

Let us consider a second order system, with complex eigenvalues
  \[ \dot{x}(t) = Ax(t) \quad (86) \]
  where $A \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^2$ and
  \[
p_A(\lambda) = \det(\lambda I_2 - A) = (\lambda - \sigma)^2 + \omega^2.
  \quad (87)\]
The two eigenvalues of \( A \) are \( \lambda_1 = \sigma + j\omega \) and \( \lambda_2 = \sigma - j\omega \), while the two eigenvectors of \( A \) are given by

\[
(\lambda_1 I_2 - A)v_1 = \Theta_2 \quad \quad (\lambda_2 I_2 - A)v_2 = \Theta_2.
\] (88)

Notice that, since \( \lambda_2 \) is the complex conjugate of \( \lambda_1 \), \( v_2 \) is the complex conjugate vector of \( v_1 \), i.e. if \( v_1 = v_r + jv_i \), then \( v_2 = v_r - jv_i \). This last fact can be verified as follows. Assume that \( \lambda_1 = \sigma + j\omega \), \( v_1 = v_r + jv_i \) and insert these expressions in the first of Eqs. (88).

\[
[(\sigma + j\omega)I_2 - A)] \begin{bmatrix} v_r + jv_i \end{bmatrix} = \Theta_2,
\]

Separating into its real and imaginary parts we obtain

\[
[\sigma v_r - \omega v_i] + j[\sigma v_i + \omega v_r] = Av_r + jAv_i
\]

\[
\begin{bmatrix} \sigma v_r - \omega v_i \\ \sigma v_i + \omega v_r \end{bmatrix} = Av.
\] (89)

Notice that Eqs. (89) hold if we replace \( \omega \) by \(-\omega\) and \( v_1 \) by \(-v_i\). Thus, if \( \lambda_1 = \sigma + j\omega \) and \( v_1 = v_r + jv_i \) are respectively an eigenvalue and eigenvector of \( A \), then \( \lambda_2 = \sigma - j\omega \) and \( v_2 = v_r - jv_i \) are also respectively an eigenvalue and eigenvector.

Eqs. (89) can be rewritten in matrix form as follows

\[
A \begin{bmatrix} v_r & v_i \end{bmatrix} = \begin{bmatrix} v_r & v_i \end{bmatrix} \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}.
\] (90)

Thus, we can define the similarity transformation matrix

\[
T = \begin{bmatrix} v_r & v_i \end{bmatrix} \in \mathbb{R}^{2 \times 2}
\] (91)

and the matrix

\[
J_c = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}
\] (92)

such that

\[
A = T J_c T^{-1}, \quad e^{At} = T e^{J_c t} T^{-1}.
\] (93)

The matrix exponential \( e^{J_c t} \) is easy to calculate. Notice that

\[
J_c = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} = \sigma I_2 + S_2
\]

where

\[
I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}
\]
is skew-symmetric, i.e. $S^T_2 = -S_2$. Thus,
$$e^{S_2 t} = \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}.$$ 

This last result can be verified by differentiating with respect to time both sides of the equation:
$$\frac{d}{dt} e^{S_2 t} = S_2 e^{S_2 t}$$

and
$$\frac{d}{dt} \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} = \begin{bmatrix} -\omega \sin(\omega t) & \omega \cos(\omega t) \\ -\omega \cos(\omega t) & -\omega \sin(\omega t) \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} = S_2 e^{S_2 t}.$$

Since $\sigma I_2 S_2 = S_2 \sigma I_2$, then
$$e^{\sigma t} = e^{\sigma t} \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}. \quad (94)$$

### 21.5.6 Examples

Consider the system
$$\dot{x}(t) = Ax(t) \quad (95)$$

where $x \in \mathbb{R}^2$ and
$$A = \begin{bmatrix} -1.3 & 1.6 \\ -1.6 & 1.3 \end{bmatrix}.$$ 

The two eigenvalues of $A$ are $\lambda_1 = j$ and $\lambda_2 = -j$. Their corresponding eigenvectors are respectively
$$v_1 = \begin{bmatrix} 0.4243 + 0.5657j \\ 0 + 0.7071j \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 0.4243 - 0.5657j \\ 0 - 0.7071j \end{bmatrix}.$$ 

Defining
$$T = \begin{bmatrix} 0.4243 & 0.5657 \\ 0 & 0.7071 \end{bmatrix} \quad (96)$$

we obtain $A T_2 = T_2 J_c$, where
$$J_c = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$
Thus,
\[ e^{Jc t} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \]
and
\[ e^{At} = T \ e^{Jc t} T^{-1} = \begin{bmatrix} 0.4243 & 0.5657 \\ 0 & 0.7071 \end{bmatrix} \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 2.3570 & -1.8856 \\ 0 & 1.4142 \end{bmatrix}. \]

Utilizing the coordinate transformation
\[ x^* = T^{-1} x \quad x = T x^*, \quad (97) \]
where \( T \) is given by Eq. (96), we can obtain from Eqs. (97) and(95)
\[ \dot{x}^*(t) = \{T^{-1}AT\} \ x^*(t) = J_c \ x^*(t). \quad (98) \]

Consider now the initial condition
\[ x(0) = \begin{bmatrix} 0.4243 \\ 0 \end{bmatrix} = T x^*(0), \quad \ x^*(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \]

The phase plot for \( x^*(t) \) is given by Fig. 8-(a), while that for \( x(t) \) is given by Fig. 8-(b).

Figure 8: Phase plots
21.6 Jordan Form

21.6.1 Motivation

In the last section, we saw that if \( A \in \mathbb{F}^{n \times n} \) has distinct eigenvalues, then there exists an invertible matrix \( T \in \mathbb{C}^{n \times n} \) such that \( T^{-1}AT \) is diagonal. In this sense, we say that \( A \) is diagonalizable by similarity transformation (the matrix \( T^{-1}AT \) is called a similarity transformation of \( A \)).

Even some matrices that have repeated eigenvalues can be diagonalized. For instance, take \( A = I_2 \). Both of the eigenvalues of \( A \) are at \( \lambda = 1 \), yet \( A \) is diagonal (in fact, any invertible \( T \) makes \( T^{-1}AT \) diagonal).

On the other hand, there are other matrices that cannot be diagonalized. Take, for instance,

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

The characteristic equation of \( A \) is \( p_A(\lambda) = \lambda^2 \). Clearly, both of the eigenvalues of \( A \) are equal to 0. Hence, if \( A \) can be diagonalized, the resulting diagonal matrix would be the 0 matrix (recall that if \( A \) can be diagonalized, then the diagonal elements are necessarily roots of the equation \( p_A(\lambda) = 0 \)). Hence, there would be an invertible matrix \( T \) such that \( T^{-1}AT = 0_{2 \times 2} \). Rearranging gives

\[
\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

Multiplying out both sides gives

\[
\begin{bmatrix} t_{21} & t_{22} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

But if \( t_{21} = t_{22} = 0 \), then \( T \) is not invertible. Hence, this \( 2 \times 2 \) matrix \( A \) cannot be diagonalized with a similarity transformation.

21.6.2 Details

It turns out that every matrix can be “almost” diagonalized using similarity transformations. Although this result is not difficult to prove, it will take us too far from the main flow of the course. The proof can be found in any decent linear algebra book. Here we simply outline the facts.
**Definition:** If $J \in \mathbb{C}^{m \times m}$ appears as

$$J = \begin{bmatrix}
\lambda & 1 & 0 & \cdots & 0 \\
0 & \lambda & 1 & \cdots & 0 \\
& \ddots & \ddots & \ddots & \\
0 & 0 & 0 & \cdots & \lambda
\end{bmatrix}$$

then $J$ is called a **Jordan** block of dimension $m$, with associated eigenvalue $\lambda$.

**Theorem:** *(Jordan Canonical Form)* For every $A \in \mathbb{F}^{n \times n}$, there exists an invertible matrix $T \in \mathbb{C}^{n \times n}$ such that

$$T^{-1}AT = \begin{bmatrix}
J_1 & 0 & \cdots & 0 \\
0 & J_2 & \cdots & 0 \\
& \vdots & \ddots & \vdots \\
0 & 0 & \cdots & J_k
\end{bmatrix}$$

where each $J_i$ is a Jordan block.

If $J \in \mathbb{F}^{m \times m}$ is a Jordan block, then using the power series definition, it is straightforward to show that

$$e^{Jt} = \begin{bmatrix}
e^{\lambda t} & te^{\lambda t} & \cdots & \frac{t^{m-1}}{(m-1)!}e^{\lambda t} \\
0 & e^{\lambda t} & \cdots & \frac{t^{m-2}}{(m-2)!}e^{\lambda t} \\
& \vdots & \ddots & \vdots \\
0 & 0 & \cdots & e^{\lambda t}
\end{bmatrix}$$

Note that if $\text{Re}(\lambda) < 0$, then every element of $e^{Jt}$ decays to zero, and if $\text{Re}(\lambda) \geq 0$, then some elements of $e^{Jt}$ diverge to $\infty$.

### 21.7 Significance

In general, there is no numerically reliable way to compute the Jordan form of a matrix. It is a conceptual tool, which, among other things, shows us the extent to which $e^{At}$ contains terms other than exponentials.

Notice that if the real part of $\lambda$ is negative, then for any finite integer $m$,

$$\lim_{t \to \infty} \{t^me^{\lambda t}\} = 0.$$ 

From this result, we see that even in the case of Jordan blocks, the signs of the real parts of the eigenvalues of $A$ determine the stability of the linear differential equation

$$\dot{x}(t) = Ax(t)$$
21.8 Problems

1. Find, by hand calculation the eigenvalues, the eigenvectors and $e^{At}$ for the matrices

(a) $A = \begin{bmatrix} 5 & -4 \\ 12 & -9 \end{bmatrix}$

(b) $A = \begin{bmatrix} 14 & -8 \\ 24 & -14 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 1 \\ -9 & -4.8 \end{bmatrix}$

(d) $A = \begin{bmatrix} -2.4 & 1.8 \\ -1.8 & -2.4 \end{bmatrix}$

2. Find (by hand calculation) the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} -3 & -2 & 2 \\ 5 & 7 & -5 \\ 5 & 8 & -6 \end{bmatrix}$$

Does $e^{At}$ have all of its terms decaying to zero?

3. Read about the MatLab command `eig` (use the MatLab manual, the Matlab primer, and/or the command >> help eig). Repeat problem 2 using `eig`, and explain any differences that you get. If a matrix has distinct eigenvalues, in what sense are the eigenvectors not unique?

4. Consider the differential equation $\dot{x}(t) = Ax(t)$, where $x(t)$ is $(3 \times 1)$ and $A$ is from problem (2) above. Suppose that the initial condition is

$$x(0) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Write the initial condition as a linear combination of the eigenvectors, and find the solution $x(t)$ to the differential equation, written as a time-dependent, linear combination of the eigenvectors.

5. Suppose that we have a 2nd order system ($x(t) \in \mathbb{R}^2$) governed by the differential equation

$$\dot{x}(t) = \begin{bmatrix} 0 & -2 \\ 2 & -5 \end{bmatrix} x(t)$$

Let $A$ denote the $2 \times 2$ matrix above.

(a) Find the eigenvalues and eigenvectors of $A$. In this problem, the eigenvectors can be chosen to have all integer entries (recall that eigenvectors can be scaled)
(b) On the grid below \((x_1/x_2 \text{ space})\), draw the eigenvectors.

(c) A plot of \(x_2(t)\) vs. \(x_1(t)\) is called a *phase-plane* plot. The variable \(t\) is not explicitly plotted on an axis, rather it is the parameter of the tick marks along the plot. On the grid above, using hand calculations, draw the solution to the equation \(\dot{x}(t) = A x(t)\) for the initial conditions

\[
\begin{align*}
x(0) &= \begin{bmatrix} 3 \\ 3 \end{bmatrix}, & x(0) &= \begin{bmatrix} -2 \\ 2 \end{bmatrix}, & x(0) &= \begin{bmatrix} -3 \\ -3 \end{bmatrix}, & x(0) &= \begin{bmatrix} 4 \\ 2 \end{bmatrix}
\end{align*}
\]

**HINT:** Remember that if \(v_i\) are eigenvectors, and \(\lambda_i\) are eigenvalues of \(A\), then the solution to \(\dot{x}(t) = A x(t)\) from the initial condition \(x(0) = \sum_{i=1}^{2} \alpha_i v_i\) is simply

\[
x(t) = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t} v_i
\]

(d) Use Matlab to create a similar picture with many (say 20) different initial conditions spread out in the \(x_1, x_2\) plane

6. Suppose \(A\) is a real, \(n \times n\) matrix, and \(\lambda\) is an eigenvalue of \(A\), and \(\lambda\) is not real, but \(\lambda\) is complex. Suppose \(v \in \mathbb{C}^n\) is an eigenvector of \(A\) associated with this eigenvalue \(\lambda\). Use the notation \(\lambda = \lambda_R + j \lambda_I\) and \(v = v_r + j v_I\) for the real and imaginary parts of \(\lambda\), and \(v\) (\(j\) means \(\sqrt{-1}\)).

(a) By equating the real and imaginary parts of the equation \(A v = \lambda v\), find two equations that relate the various real and imaginary parts of \(\lambda\) and \(v\).

(b) Show that \(\bar{\lambda}\) (complex conjugate of \(\lambda\)) is also an eigenvalue of \(A\). What is the associated eigenvector?
(c) Consider the differential equation $\dot{x} = Ax$ for the $A$ described above, with the eigenvalue $\lambda$ and eigenvector $v$. Show that the function

$$x(t) = e^{\lambda t} \left[ \cos(\lambda t) v_R - \sin(\lambda t) v_I \right]$$

satisfies the differential equation. What is $x(0)$ in this case?

(d) Fill in this sentence: If $A$ has complex eigenvalues, then if $x(t)$ starts on the __________ part of the eigenvector, the solution $x(t)$ oscillates between the __________ and __________ parts of the eigenvector, with frequency associated with the __________ part of the eigenvalue. During the motion, the solution also increases/decreases exponentially, based on the __________ part of the eigenvalue.

(e) Consider the matrix $A$

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

It is possible to show that

$$A \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ j\frac{1}{\sqrt{2}} & -j\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ j\frac{1}{\sqrt{2}} & -j\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 + 2j & 0 \\ 0 & 1 - 2j \end{bmatrix}$$

Sketch the trajectory of the solution $x(t)$ in $\mathbb{R}^2$ to the differential equation $\dot{x} = Ax$ for the initial condition $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(f) Find $e^{At}$ for the $A$ given above. NOTE: $e^{At}$ is real whenever $A$ is real. See the notes for tricks in making this easy.

7. Consider the 2-state system governed by the equation $\dot{x}(t) = Ax(t)$. Shown below are the phase-plane plots ($x_1(t)$ vs $x_2(t)$) for 4 different cases. Match the plots with the $A$ matrices, and correctly draw in arrows indicating the evolution in time. Put your answers on the enlarged graphs included in the solution packet.

$$A_1 = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$$