22 Frequency Response for Linear Systems

22.1 Theory for Stable System: Complex Input Signal

Consider the linear dynamical system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{99}$$

We assume that there are n states, m inputs, and q outputs (so $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{p\times n}$, $D \in \mathbf{R}^{p\times m}$).

If the system is stable, (ie., all of A's eigenvalues have negative real parts) it is "intuitively" clear that if u is a sinusoid, then y will approach a steady-state behavior that is sinusoidal, at the same frequency, but with different amplitude and phase. In this section, we make this idea precise.

Take $\omega \geq 0$ as the input frequency, and (although not physically relevant) let $\bar{u} \in \mathbf{C}^m$ be a fixed complex vector. Take the input function $u(t)$ to be

$$u(t) = \bar{u}e^{j\omega t}$$

for $t \geq 0$. Then, the response is

$$\begin{aligned} x(t) &= e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\ &= e^{At}x_0 + eAt \int_0^t e^{-A\tau}B\bar{u}e^{j\omega\tau}d\tau \\ &= e^{At}x_0 + e^{At}\int_0^t e^{(j\omega I - A)\tau}d\tau B\bar{u} \end{aligned}$$

Now, since A is stable, all of the eigenvalues have negative real parts. This means that all of the eigenvalues of $(j\omega I - A)$ have positive real parts. Hence $(j\omega I - A)$ is invertible, and we can write the integral as

$$\begin{aligned} x(t) &= e^{At}x_0 + e^{At}\int_0^t e^{(j\omega I - A)\tau}d\tau B\bar{u} \\ &= e^{At} + e^{At}\left(j\omega I - A\right)^{-1} e^{(j\omega I - A)\tau}\Big|_0^t \; B\bar{u} \\ &= e^{At}x_0 + e^{At}\left(j\omega I - A\right)^{-1}\left[e^{(j\omega I - A)t} - I\right]B\bar{u} \\ &= e^{At}x_0 + e^{At}\left[e^{(j\omega I - A)t} - I\right]\left(j\omega I - A\right)^{-1}B\bar{u} \\ &= e^{At}x_0 + e^{At}\left[e^{j\omega t}e^{At} - I\right]\left(j\omega I - A\right)^{-1}B\bar{u} \\ &= e^{At}x_0 + \left[e^{j\omega t} - e^{At}\right]\left(j\omega I - A\right)^{-1}B\bar{u} \\ &= e^{At}\left[x_0 - \left(j\omega I - A\right)^{-1}B\bar{u}\right] + \left(j\omega I - A\right)^{-1}B\bar{u}e^{j\omega t} \end{aligned}$$

Hence, the output $y(t)$ would satisfy

$$y(t) = Ce^{At}\left[x_0 - \left(j\omega I - A\right)^{-1}B\bar{u}\right] + C\left(j\omega I - A\right)^{-1}B\bar{u}e^{j\omega t} + D\bar{u}e^{j\omega t}$$

In the limit as $t \to \infty$, the first term decays to 0 exponentially, leaving the steady-state response

$$y_{ss}(t) = \left[D + C\left(j\omega I - A\right)^{-1}B\right]\bar{u}e^{j\omega t}$$

Hence, we have verified our initial claim – if the input is a complex sinusoid, then the steady-state output is a complex sinusoid at the same exact frequency, but amplified by a complex gain of $D + C\,(j\omega I - A)^{-1}\,B$.

The function $G(j\omega)$

$$G(j\omega) := D + C\,(j\omega I - A)^{-1}\,B \tag{100}$$

is called the frequency response of the linear system in (99). Hence, for stable systems, we have proven

$$u(t) := \bar{u}e^{j\omega t} \quad \Rightarrow \quad y_{ss}(t) = G(j\omega)\bar{u}e^{j\omega t}$$

More precisely, this is the frequency response from u to y, so we might also write $G_{yu}(j\omega)$ to indicate what is the input (u) and what is the output (y). G_{yu} can be calculated rather easily using a computer, simply by evaluating the matrix expression in (100) at a large number of frequency points $\omega \in \mathbf{R}$.

22.2 MIMO Systems: Response due to real sinusoidal inputs

In the case where the system has multiple inputs and outputs, it is a bit more complicated to write out the response due to sinusoidal inputs at a fixed frequency, since the different inputs may all have different magnitudes and phases. Suppose that there are m inputs, and q outputs (so $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{p\times n}$, $D \in \mathbf{R}^{p\times m}$). Take $a \in \mathbf{R}^m$, $b \in \mathbf{R}^m$, and $\omega \geq 0$. Consider the input

$$u(t) = a\cos\omega t \; + \; b\sin\omega t$$

Note that this is the real part of a complex input, namely

$$u(t) = \text{Re}\left[(a - jb)e^{j\omega t}\right]$$

Hence, the steady state output must be the real part of a function, specifically,

$$y(t) = \text{Re}\left[G(j\omega)(a - jb)e^{j\omega t}\right]$$

So, in summary: To determine the steady-state response due to an input

$$u(t) = a\cos\omega t \; + \; b\sin\omega t,$$

let $c \in \mathbf{R}^q$ and $d \in \mathbf{R}^q$ be real vectors so that

$$c - jd = G(j\omega)(a - jb)$$

Then, the steady-state output is

$$y_{ss}(t) = c\cos\omega t \; + \; d\sin\omega t$$

22.3 Experimental Determination

Since the frequency response has the interpretation as a representation of the frequency-dependent amplitude gain and phase shift between a sinusoidal input and steady-state output, it is easy to obtain experimentally for a given physical system. This can be done by performing several different forced-response experiments, with the forcing being a sinusoid, each experiment using a different frequency.