5 Simple Cruise-Control

The car model that will consider in this chapter is made up of:

- An accurate positioning motor, with power amplifier (we’ll see how to make these later in the course) that accepts a low-power (almost no current) voltage signal, and moves the accelerator pedal in a manner proportional to the voltage signal.

- The engine, which produces a torque that is related to the position of the accelerator pedal (current and past values).

- The drivetrain of the car, which transmits this torque to the ground through the driven wheels. We will assume that the car is also subjected to air and rolling resistance.

- A vehicle, which gets accelerated due to the forces (road/tire interface, aerodynamic) which act on it.

- Changes in the slope of the highway, which act as a disturbance force on the car. We do not assume that the car has a sensor which can measure this slope.

- A speedometer, which converts the speed (in miles/hour, say), into a voltage signal for feedback purposes. We will assume for now that we get 1 volt for every mile/hour. Hence, the variables \( v_{\text{meas}} \) and \( v \) are related by

\[
v_{\text{meas}}(t)\text{[volts]} = v(t)\text{[miles/hour]}.
\]

A diagram for the system is shown in Fig. 2.
5.1 Control System

5.1.1 Open-Loop Control

As stated previously, in the case of open-loop control, the control system is designed to be an inverse of the plant. For example, if the relationship between the control input $u$ and the car velocity $v$ is given by a static nonlinearity $P(\cdot)$, so that at every $t$

$$v(t) = P(u(t))$$

and the desired velocity is $v_{\text{des}}(t)$, then the open-loop control actions will be

$$u(t) = P^{-1}(v_{\text{des}}(t)),$$

where $P^{-1}(\cdot)$ is the inverse of the function $P(\cdot)$. For a level road,

$$v(t) = P(u(t)) = P \left( P^{-1}(v_{\text{des}}(t)) \right) = v_{\text{des}}(t)$$

which is perfect.

The biggest problem with an open-loop control systems is that it relies totally on calibration, and cannot effectively deal with external disturbances (road grade) or plant uncertainties (aerodynamic variations, condition of engine system, tire inflation). We will elaborate on these points later.

5.1.2 Closed-Loop Control

In closed-loop control, the voltage generated by the control system depends on the measured speed, in addition to the desired speed. A block diagram is shown below. Note that through feedback, the effect of the hill is somewhat compensated for. For instance, if the system is operating correctly on a flat road, with $v(t) = v_{\text{des}}(t) = \bar{v}$, then if the car encounters a hill the following occurs:

- Because of the hill, the speed of the car changes
- The controller, measuring the actual speed, and the desired speed, detects an error in the speed regulation
- The controller compensates for the error by adjusting the output voltage, which moves the throttle into a new position, compensating for the speed error

In a fully automated system, the brakes would also have to be used.
5.2 Steady State Conditions

In this example, we want to look at the cruise control system in the **steady state**. We will assume the following (all of which are realistic, but not guaranteed - see problem 10, for a counterexample, and problem 11 for a converse point-of-view) for each of the subsystems (car and controller) as well as a similar assumption for the closed-loop behavior.

**For the car:** If the voltage (to accelerator position device) is held constant and the slope of the hill is constant, the car eventually reaches a steady-state speed. This steady-state speed is only a function of the voltage and hill-grade. It is not dependent on the past history.

**For the controller:** If the signals into the controller go to constant values, the output voltage of the controller eventually reaches a steady-state value. Again, this steady-state value depends only on the steady-state values of the signals going into the controller.

**For the closed-loop:** If the desired-velocity input to the controller is held fixed and the slope of the hill is fixed, the car eventually reaches a steady-state speed, and the output-voltage from the controller goes to a steady-state value.

In this subsection, we will see how the **steady-state behavior of the controlled system** can be determined in terms of the **steady-state properties of each individual subsystem**.

For simplicity, we are going to assume that the **steady-state** value of the car’s speed has the following properties

1. On a flat road, if \( u = 0 \), then \( v = 0 \)
2. A unit change in $u$ causes $P$ change in $v$. If the units of $u$ are "volts" and the units of $v$ are "mph", then the units of the number $P$ are "mph/volt." For example, $P = 10\text{ mph/volt}$.

3. Using $\gamma$ to denote the road-grade angle, the force due to gravity along the direction of travel is $-mg \sin \gamma(t)$. Define $w(t) := 100 \sin \gamma(t)$ so that $w$ represents the "percent inclination" which could vary between -100 and 100 (more realistically between -10 and 10). So, $w = 1$ corresponds to an uphill with 1 meter rise for every 100 meters driven. We assume that each unit change in $w$ causes $-5 \text{ mph}$ change in $v$.

4. Superposition holds.

5. The measured speed is numerically equal to the actual speed, i.e. $v_{\text{meas}} = v$.

Hence, our steady-state model for the actuator/engine/drivetrain/car/hill is simply

$$v = Pu + Dw,$$

where $P = 10 \text{ mph/volt}$ and $D = -5 \text{ mph}$.

### 5.2.1 Open-Loop Control

Assuming a level road (i.e. $w = 0$), we need to design a controller which will invert the plant, without measuring $v$.

We assume that the controller steady-state behavior is given by a linear relationship, $u = K v_{\text{des}}$, as shown in Fig. 4. Using $w = 0$ as the design point, a good choice for the constant $K$ is $K := P^{-1} = 1/10$, which results in the controlled system behavior

$$v = v_{\text{des}} + Dw.$$

Notice that if we have a 1\% grade (i.e. $w = 1$), then

$$e = v_{\text{des}} - v = -D = 5 \text{ mph}$$

The open-loop controller does not compensate for the hill in any manner.
For the controller, we will assume that the steady-state behavior of the control system logic is

\[ u = K_{ff} v_{des} - K_{fb} v \]  

as shown in Fig. 5. What is the steady-state behavior of the controlled (closed-loop) system? To determine this, we need to eliminate \( u \) from equations (15) and (16), and derive a relationship between \( v_{des} \), \( w \) and \( v \). This is simple to do. Note that

\[ v = Pu + Dw \]

\[ = P [K_{ff} v_{des} - K_{fb} v] + Dw \].

Moving all \( v \) terms to one side gives

\[ [1 + PK_{fb}] v = PK_{ff} v_{des} + Dw \],

which can be solved for \( v \) simply by dividing. This yields

\[ v = \frac{PK_{ff}}{1 + PK_{fb}} v_{des} + \frac{D}{1 + PK_{fb}} w \].  

(17)

We need to find adequate values for the control gains \( K_{ff} \) and \( K_{fb} \). To do this, recall that the goal is \( v \approx v_{des} \), in spite of the road grade, and possible variations in the car \( (P) \) itself.

From Eq. (17) we see that the control gains should be adjusted so that

\[ \frac{PK_{ff}}{1 + PK_{fb}} \approx 1 \]  

(18)

and

\[ \left| \frac{D}{1 + PK_{fb}} \right| < |D| \].  

(19)
From Eq. (19) we see that we need $|1 + PK_{fb}| > 1$. In fact (from this steady state point of view) the larger $|1 + PK_{fb}|$, the better. This can be accomplished by setting

$$|PK_{fb}| >> 1. \tag{20}$$

which is referred to as a high feedback loop gain. Assuming that equation (20) applies, equation (18) will be satisfied if $K_{ff} \approx K_{fb}$. In our numerical example, $P = 10$. Thus, using $K_{fb} = 1$ and $K_{ff} = 1.1$, we obtain

$$v = v_{des} + \frac{-5}{11}w.$$  

In this case, a 1% grade (i.e. $w = 1$) will cause $e = v_{des} - v = \frac{5}{11} \approx 0.5 \text{ mph}$ velocity error, which is much smaller than the 5 mph error obtained with the open-loop controller.

### 5.2.3 Sensitivity Analysis

We now consider the sensitivity of both open-loop and close-loop control to small changes in the plant parameters. Assume that the control system is designed for a nominal plant value of $P_{\text{nom}}$ but that the actual plant is

$$P = P_{\text{nom}} + \Delta.$$ 

Let us now define the transfer function $T(P)$ which relates the actual velocity $v$ to the desired velocity $v_{des}$, for a given plant $P$. For closed-loop control, the transfer function is given by

$$T_{cl}(P) := \frac{PK_{ff}}{1 + PK_{fb}}.$$ 

For open-loop control, the transfer function is given by

$$T_{ol}(P) := PK.$$ 

Now defining the fractional uncertainty change $\%P$ by

$$\%P := \frac{P - P_{\text{nom}}}{P_{\text{nom}}} = \frac{\Delta}{P_{\text{nom}}},$$

define and calculate the sensitivity function,

$$S := \frac{\%T(P)}{\%P} = \frac{T(P) - T(P_{\text{nom}})}{T(P_{\text{nom}})} = \frac{P_{\text{nom}}}{T(P_{\text{nom}})} \frac{T(P_{\text{nom}} + \Delta) - T(P_{\text{nom}})}{\Delta}. \tag{21}$$

for both the open-loop and closed-loop cases. In the limit of small deviations $\Delta$, equation (21) becomes

$$S = \left. \frac{\%T(P)}{\%P} = \frac{P_{\text{nom}}}{T(P_{\text{nom}})} \frac{dT(P)}{dP} \right|_{P=P_{\text{nom}}}. $$
For the open-loop system we obtain:

\[
\frac{dT_{ol}(P)}{dP} \bigg|_{P=P_{\text{nom}}} = K \quad T_{ol}(P_{\text{nom}}) = KP_{\text{nom}}
\]

\[
S_{ol} = \frac{P_{\text{nom}}}{KP_{\text{nom}}} K = 1.
\] (22)

For the closed-loop system we obtain:

\[
\frac{dT_{cl}(P)}{dP} \bigg|_{P=P_{\text{nom}}} = \frac{K_{ff}}{1 + P_{\text{nom}}K_{fb}} - \frac{P_{\text{nom}}K_{ff}K_{fb}}{(1 + P_{\text{nom}}K_{fb})^2}
\]

\[
= \frac{K_{ff}}{(1 + P_{\text{nom}}K_{fb})^2}
\] (23)

and

\[
S_{cl} = \frac{P_{\text{nom}}}{K_{ff}P_{\text{nom}}}rac{K_{ff}}{1 + P_{\text{nom}}K_{fb}} (1 + P_{\text{nom}}K_{fb})^2
\]

\[
= \frac{1}{1 + P_{\text{nom}}K_{fb}}.
\] (24)

Thus, comparing both the open-loop with the closed-loop sensitivity transfer functions, we see that we can make the closed-loop sensitivity much smaller by having a high feedback loop gain.

As an example, assume that \(P_{\text{nom}} = 10\), \(P = 9\), so \(\%P = 10\%\). Then we have

\[
S_{ol} = 1 \implies \%T_{ol} \approx 10\%
\]

while

\[
S_{cl} \approx \frac{1}{11} \implies \%T_{cl} = S_{cl}\%P \approx 1\%.
\]

From the discussion in these two sections, it is apparent that in order to have a small closed-loop control sensitivity to external disturbances and plant uncertainties, it is necessary to have high feedback loop gains. Unfortunately, high feedback loop gain can lead to instability, when the control plant is dynamic. To understand the effect of dynamics in the response of the control system, we will now use a simple dynamic model for the car.

### 5.3 Dynamic Model

There are 3 forces acting on the car:
• $F_{\text{eng}}(t)$, the static friction force at wheel/road interface, due to the torque applied to the driveshaft from engine.

• $F_{\text{drag}}$, Aerodynamic drag forces, which depend on the speed of the car.

• $d_F(t)$ disturbance force, for instance the gravitational force due to inclination in the road, or additional aerodynamic forces due to wind gusts. For now we’ll take $d_F = -0.01mgw = Gw$, with $G = -0.01mg$, where $w$ is the percent inclination, defined as before, $w(t) := 100 \sin \gamma(t)$.

We will model the engine and aero forces as follows:

1. There is a DC motor which controls the throttle position. We assume that the engine force is related to the voltage signal to the DC motor by a constant of proportionality, $E$, hence

   $$F_{\text{eng}}(t) = Eu(t)$$

   where $u(t)$ is the voltage applied to the DC motor.

2. For simplicity, we assume that the aerodynamic force is proportional to the speed of the car, hence

   $$F_{\text{drag}}(t) = -\alpha v(t)$$

   where $\alpha$ is a positive constant.

Hence, the differential equation relating $u$, $d_F$ (or $w$) and $v$ is simply

$$m \dot{v}(t) = -\alpha v(t) + Eu(t) + d_F(t)$$

$$= -\alpha v(t) + Eu(t) + Gw(t)$$

(25)

Eq. (25) is a first order, Linear, Time Invariant (LTI) Ordinary Differential Equation (ODE). In order to continue our discussion, we will first review the solution of first order LTI ODEs.

5.4 Solution of a First Order LTI ODE

Consider the following system

$$\dot{x}(t) = ax(t) + bu(t)$$

(26)

where $u$ is the input, $x$ is the dependent variable, and $a$ and $b$ are constant coefficients. Given the initial condition $x(0) = x_o$ and an arbitrary input function $u(t)$ defined on $[0, \infty)$, the solution of Eq. (26) is given by

$$x_s(t) = e^{at} x_o + \int_0^t e^{a(t-\tau)} b u(\tau) d\tau$$

(27)
You can derive this with the integrating factor method. We can also easily just check that (27) is indeed the solution of (26) by verifying two facts: the function \( x_s \) satisfies the differential equation for all \( t \geq 0 \); and \( x_s \) satisfies the given initial condition at \( t = 0 \). The theory of differential equations tells us that there is one and only one function that satisfies both (uniqueness of solutions). To verify, first check the value of \( x_s(t) \) at \( t = 0 \):

\[
x_s(0) = e^{a0} x_o + \int_0^0 e^{a(t-\tau)} b u(\tau) \, d\tau = x_o.
\]

Taking the time derivative of (27) we obtain

\[
\dot{x}_s(t) = a e^{at} x_o + \frac{d}{dt} \left\{ e^{at} \int_0^t e^{-a\tau} b u(\tau) \, d\tau \right\} \\
= a e^{at} x_o + a e^{at} \int_0^t e^{-a\tau} b u(\tau) \, d\tau + e^{at} e^{-at} b u(t) \\
= a x_s(t) + b u(t).
\]

as desired.

### 5.4.1 Free response

Fig. 6 shows the normalized free response (i.e. \( u(t) = 0 \)) of the solution Eq. (27) of ODE (26) when \( a < 0 \). Notice that the slope at time zero is \( \dot{x}(0) = ax_o \) and \( T = 1/|a| \) is the time

![Figure 6: Normalized Free response of first order system \((a < 0)\)](image-url)
that \( x(t) \) would cross 0 if the initial slope is continued, as shown in the figure. The time
\[
T := \frac{1}{|a|}
\]
is called the \textit{time constant} of a first order asymptotically stable system (\( a < 0 \)). \( T \) is an indication of how fast is the response of the system. The larger \(|a|\), the smaller \( T \) and the faster the response of the system.

Notice that
\[
\frac{x_{\text{free}}(T)}{x_0} = \frac{1}{e} \approx .37 = 37\%
\]
\[
\frac{x_{\text{free}}(2T)}{x_0} = \frac{1}{e^2} \approx .13 \approx 10\%
\]
\[
\frac{x_{\text{free}}(3T)}{x_0} = \frac{1}{e^3} \approx .05 = 5\%
\]

If \( a > 0 \) the free response of ODE (26) is \textit{unstable}, i.e. \( \lim_{t \to \infty} |x(t)| = \infty \). When \( a = 0 \), \( x(t) = x_0 \) and we say that this system is \textit{limitedly stable}.

### 5.4.2 Forced response

We first consider the system response to a \textit{step input}. In this case, the input \( u(t) \) is given by
\[
u(t) = u_m \mu(t) = \begin{cases} 0 & \text{if } t < 0 \\ u_m & \text{if } t \geq 0 \end{cases}
\]
where \( u_m \) is a constant and \( x(0) = 0 \). The solution (27) yields
\[
x(t) = \frac{b}{-a} \left(1 - e^{at}\right) u_m.
\]
If \( a < 0 \), the steady state output \( x_{ss} \) is \( x_{ss} = \frac{b}{-a} u_m \).

It is easy to show that when \( a < 0 \), then if \( u(t) \) is uniformly (in time) bounded by a positive number \( M \), then the resulting solution \( x(t) \) will be uniformly bounded by \( \frac{bM}{|a|} \). To derive this, suppose that \( |u(\tau)| \leq M \) for all \( \tau \geq 0 \). Then for any \( t \geq 0 \), we have
\[
|x(t)| = \left| \int_0^t e^{a(t-\tau)} b u(\tau) \, d\tau \right|
\]
\[
\leq \int_0^t \left| e^{a(t-\tau)} b u(\tau) \right| \, d\tau
\]
\[
\leq \int_0^t e^{a(t-\tau)} b M \, d\tau
\]
\[
\leq \frac{bM}{-a} \left(1 - e^{at}\right)
\]
\[
\leq \frac{bM}{-a}.
\]
Thus, if \( a < 0 \), \( x(0) = 0 \) and \( u(t) \leq M \), the output is bounded by \( x(t) \leq |bM/a| \). This is called a \textit{bounded-input, bounded-output} (BIBO) system. If the initial condition is non-zero, the output \( x(t) \) will still be bounded since the magnitude of the free response monotonically converges to zero, and the response \( x(t) \) is simply the sum of the free and forced responses. \textbf{Note:} The system is \textbf{not} bounded-input/bounded-output when \( a \geq 0 \).

### 5.4.3 Sinusoidal Response

Consider the linear dynamical system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]  

(28)

We assume that \( A, B, C \) and \( D \) are scalars \((1 \times 1)\).

If the system is stable, (i.e., \( A < 0 \)) it is “intuitively” clear that if \( u \) is a sinusoid, then \( y \) will approach a steady-state behavior that is sinusoidal, at the same frequency, but with different amplitude and phase. In this section, we make this idea precise.

Take \( \omega \geq 0 \) as the input frequency, and (although not physically relevant) let \( \bar{u} \) be a fixed complex number and take the input function \( u(t) \) to be

\[ u(t) = \bar{u}e^{j\omega t} \]

for \( t \geq 0 \). Then, the response is

\[
\begin{align*}
x(t) &= e^{At}x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \\
&= e^{At}x_0 + e^{At} \int_0^t e^{-A\tau} B\bar{u}e^{j\omega \tau} d\tau \\
&= e^{At}x_0 + e^{At} \int_0^t e^{(j\omega - A)\tau} d\tau B\bar{u}
\end{align*}
\]  

(29)

Now, since \( A < 0 \), regardless of \( \omega \), \( (j\omega - A) \neq 0 \), and we can solve the integral as

\[
x(t) = e^{At} \left[ x_0 - \frac{B\bar{u}}{j\omega - A} \right] + \frac{B}{j\omega - A} e^{j\omega t}
\]  

(30)

Hence, the output \( y(t) = Cx(t) + Du(t) \) would satisfy

\[
y(t) = Ce^{At} \left[ x_0 - \frac{B\bar{u}}{j\omega - A} \right] + \left[ D + \frac{CB}{j\omega - A} \right] \bar{u}e^{j\omega t}
\]

In the limit as \( t \to \infty \), the first term decays to 0 exponentially, leaving the steady-state response

\[
y_{ss}(t) = \left[ D + \frac{CB}{j\omega - A} \right] \bar{ue}^{j\omega t}
\]

Hence, we have verified our initial claim – if the input is a complex sinusoid, then the steady-state output is a complex sinusoid at the same exact frequency, but “amplified” by a complex gain of \( \left[ D + \frac{CB}{j\omega - A} \right] \).
The function $G(j\omega)$
\begin{equation}
G(j\omega) := D + \frac{CB}{j\omega - A}
\end{equation}

is called the frequency response of the system in (99). Hence, for stable, first-order systems, we have proven
\begin{equation}
\tilde{u}(t) := \tilde{u}e^{j\omega t} \Rightarrow y_{ss}(t) = G(j\omega)\tilde{u}e^{j\omega t}
\end{equation}

$G$ can be calculated rather easily using a computer, simply by evaluating the expression in (100) at a large number of frequency points $\omega \in \mathbb{R}$.

What is the meaning of a complex solution to the differential equation (99)? Suppose that functions $u, x$ and $y$ are complex, and solve the ODE. Denote the real part of the function $u$ as $u_R$, and the imaginary part as $u_I$ (similar for $x$ and $y$). Then, for example, $x_R$ and $x_I$ are real-valued functions, and for all $t$ $x(t) = x_R(t) + jx_I(t)$. Differentiating gives
\begin{equation}
\frac{dx}{dt} = \frac{dx_R}{dt} + j\frac{dx_I}{dt}
\end{equation}

Hence, if $x, u$ and $y$ satisfy the ODE, we have (dropping the $(t)$ argument for clarity)
\begin{align*}
\frac{dx_R}{dt} + j\frac{dx_I}{dt} &= A(x_R + jx_I) + B(u_R + ju_I) \\
y_R + jy_I &= C(x_R + jx_I) + D(u_R + ju_I)
\end{align*}

But the real and imaginary parts must be equal individually, so exploiting the fact that the coefficients $A, B, C$ and $D$ are real numbers, we get
\begin{align*}
\frac{dx_R}{dt} &= Ax_R + Bu_R \\
y_R &= Cx_R + Du_R
\end{align*}

and
\begin{align*}
\frac{dx_I}{dt} &= Ax_I + Bu_I \\
y_I &= Cx_I + Du_I
\end{align*}

Hence, if $(u, x, y)$ are functions which satisfy the ODE, then both $(u_R, x_R, y_R)$ and $(u_I, x_I, y_I)$ also satisfy the ODE.

Finally, we need to do some trig/complex number calculations. Suppose that $H \in \mathbb{C}$ is not equal to zero. Recall that $\angle H$ is the real number (unique to within an additive factor of $2\pi$) which has the properties
\begin{align*}
\cos \angle H &= \frac{\text{Re}H}{|H|}, & \sin \angle H &= \frac{\text{Im}H}{|H|}
\end{align*}
Then,
\[
\text{Re} (He^{j\theta}) = \text{Re} [(H_R + jH_I) (\cos \theta + j \sin \theta)] \\
= H_R \cos \theta - H_I \sin \theta \\
= |H| \left[ \frac{H_R}{|H|} \cos \theta - \frac{H_I}{|H|} \sin \theta \right] \\
= |H| \left[ \cos \angle H \cos \theta - \sin \angle H \sin \theta \right] \\
= |H| \cos (\theta + \angle H)
\]

\[
\text{Im} (He^{j\theta}) = \text{Im} [(H_R + jH_I) (\cos \theta + j \sin \theta)] \\
= H_R \sin \theta + H_I \cos \theta \\
= |H| \left[ \frac{H_R}{|H|} \sin \theta + \frac{H_I}{|H|} \cos \theta \right] \\
= |H| \left[ \cos \angle H \sin \theta + \sin \angle H \cos \theta \right] \\
= |H| \sin (\theta + \angle H)
\]

Now consider the differential equation/frequency response case. Let \( G(\omega) \) denote the frequency response function. If the input \( u(t) = \cos \omega t = \text{Re} (e^{j\omega t}) \), then the steady-state output \( y \) will satisfy
\[
y(t) = |G(\omega)| \cos (\omega t + \angle G(\omega))
\]
A similar calculation holds for \( \sin \), and these are summarized below.

<table>
<thead>
<tr>
<th>Input</th>
<th>Steady-State Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( G(0) = D - \frac{C}{\alpha} )</td>
</tr>
<tr>
<td>( \cos \omega t )</td>
<td>(</td>
</tr>
<tr>
<td>( \sin \omega t )</td>
<td>(</td>
</tr>
</tbody>
</table>

### 5.5 Control of the First Order Car Model

We can now return to analyzing both the open-loop and closed-loop response of the cruise control system, when the car is modeled by the first order ODE in (25). Suppose that \( u(t) \) is constant, at \( \bar{u} \), and \( w(t) \) is constant, \( \bar{w} \). The steady-state velocity of the car is just
\[
\bar{v} = \frac{E}{\alpha} \bar{u} + \frac{G}{\alpha} \bar{w}
\]
which is of the simple form (with \( P := \frac{E}{\alpha}, D := \frac{G}{\alpha} \)) as proposed back in equation (15).

#### 5.5.1 Open-Loop Control System

In the absence of a disturbance force \( (w(t) \equiv 0) \), the steady-state speed of the car due to a constant input voltage \( u(t) \equiv \bar{u} \) is
\[
\bar{v} := \lim_{t \to \infty} v(t) = \frac{E}{\alpha} \bar{u}
\]
Hence, a feasible Open-Loop control system would invert this relationship, namely

\[ u(t) = \frac{\alpha}{E} v_{\text{des}}(t) \]

Here, we let \( K_{ol} := \frac{\alpha}{E} \) is the open-loop controller gain. Note that this gain must be implemented in the control circuitry, and hence must be computed using values of \( E \) and \( \alpha \) that are believed accurate.

The equations which govern the open-loop car are now

\[ \dot{v}(t) = -\frac{\alpha}{m} v(t) + \frac{E K_{ol}}{m} v_{\text{des}}(t) + \frac{G}{m} w(t). \] (32)

### 5.5.2 Closed-Loop Control System

We now consider a closed-loop control system. The feedback controller will be a circuit, which combines the desired speed voltage (which is also in the same units – 1 volt for every mile/hour) and the actual speed voltage to produce a control voltage, which commands the throttle positioning DC motor system. For simplicity, take the control law to be a "constant gain" control law similar to Eq. (16), which was used for the steady state model.

\[ u(t) = K_{ff} v_{\text{des}}(t) - K_{fb} v(t) \]

The block diagram of the control system is shown in Fig. 7

![Figure 7: Cruise Closed-Loop Control System](image)

Plugging this control law into the car model (25) gives

\[ \dot{v}(t) = -\frac{\alpha}{m} v(t) + \frac{E}{m} [K_{ff} v_{\text{des}}(t) - K_{fb} v(t)] + \frac{G}{m} w(t) \]

\[ = -\frac{\alpha}{m} v(t) + \frac{E K_{fb}}{m} v(t) + \frac{E K_{ff}}{m} v_{\text{des}}(t) + \frac{G}{m} w(t). \] (33)
5.5.3 Open/Closed Loop Comparison

Equations (32) and (33) respectively govern the behavior of the car under open-loop and closed-loop control. We will compare the performance, using the subscript \( \_\text{ol} \) to refer to open-loop control, and \( \_\text{cl} \) to refer to closed-loop control.

First notice that both are systems are governed by first-order linear ODEs with constant coefficients, similar to equation (26).

First, the time constants are different,

\[
T_{\text{ol}} = \frac{m}{\alpha}, \quad T_{\text{cl}} = \frac{m}{\alpha + E K_{fb}}.
\]

For positive \( K_{fb} \), the response of the closed-loop system is always faster than that of the open-loop system.

Now assume that both the desired velocity and the incline disturbance are both constants, respectively given by \( \bar{v}_{\text{des}} \), \( \bar{d}_F \). The resulting steady-state speeds (in open and closed-loop) are

\[
\bar{v}_{\text{ol}} = \bar{v}_{\text{des}} + \frac{1}{\alpha} \bar{d}_F, \quad \bar{v}_{\text{cl}} = \frac{E K_{ff}}{\alpha + E K_{fb}} \bar{v}_{\text{des}} + \frac{1}{\alpha + E K_{fb}} \bar{d}_F
\]

As in the earlier steady state analysis, by selecting \( K_{fb} \) sufficiently large, and setting

\[
\frac{E K_{ff}}{\alpha + E K_{fb}} \approx 1
\]

we can achieve better disturbance rejection and tracking with closed-loop control than with open-loop control.

Do a numerical simulation to study the responses of both the open-loop and closed-loop control system. For the closed-loop system, take \( K_{fb} := 8, K_{ff} := 9.5 \).

Assume the following numerical values for the parameters: \( m = 1000, \alpha = 60, E = 40 \) and \( G = -98 \), and assume the following conditions: \( v(0) = 25 \).

\[
v_{\text{des}}(t) = \begin{cases} 
25 & \text{if } 0 < t < 10 \\
30 & \text{if } t \geq 10 
\end{cases}
\]

\[
w(t) = \begin{cases} 
0 & \text{if } t < 80 \\
5 & \text{if } t \geq 80 
\end{cases}
\]

The time response of the two control systems is shown below.
The control signal $u(t)$ is shown in the figure below.

$$m\ddot{v}(t) = -\alpha v(t) + Eu(t) + Gw(t)$$
For a given constant value of \( \bar{v}_{\text{des}}, \bar{w} \), the ideal steady-state input \( \bar{u} \) is
\[
\bar{u} := \frac{\alpha}{E} \bar{v}_{\text{des}} - \frac{G}{\alpha} \bar{w}
\]
This is the only possible value for the steady-state value of \( u \) which gives a steady-state velocity exactly equal to the desired velocity. Note that the “correct” control depends on both \( \bar{v}_{\text{des}} \) and \( \bar{w} \).

However, with our current closed-loop control law, if the car has speed \( v(t) = \bar{v}_{\text{des}} \), then the control effort is
\[
u(t) = (K_{ff} - K_{fb}) \bar{v}_{\text{des}}
\]
With the proportional control, the car stops accelerating when the force from the controller signal balances the force from air resistance and the incline.

The commanded force from the controller, with \( v_{\text{des}} \) fixed, is a linear (actually linear+offset, called affine) function of \( v \). Hence, it is easy to see that for different values of \( w \), the car will cease accelerating (i.e. steady-state) for values of \( v \) which are different from \( v_{\text{des}} \). The “problem” is that the instantaneous throttle position is simply a function of the instantaneous desired speed and actual speed. There is no mechanism for the throttle to be opened further if this error in speed does not eventually decrease.

A new strategy should incorporate a term that corresponds to “if the speed is below the desired speed, increase the throttle.” A mathematical version of this statement is given below:
\[
\dot{u}(t) = K_I [v_{\text{des}}(t) - v(t)]
\]
Note that by this strategy, at any time \( t \), for which \( v_{\text{des}}(t) > v(t) \), the control law is, at that instant, increasing the throttle angle. Likewise, at any time \( t \), for which \( v_{\text{des}}(t) < v(t) \), the control law is, at that instant, decreasing the throttle angle. Finally, at any time \( t \), for which \( v_{\text{des}}(t) = v(t) \), the control law is, at that instant, not changing the throttle angle.

Integrating from \( t = 0 \) to \( t \) gives
\[
u(t) = u(0) + K_i \int_0^t [v_{\text{des}}(\xi) - v(\xi)] d\xi
\]
which shows that the control “action,” \( u(t) \), is a running integral of the error between the desired value and the actual value. This is called integral control action. As you would expect, integral control can be used to reduce the steady-state error due to hills.

A block diagram of the closed-loop system, using integral control is shown on the board.
The equation for the car is as before

\[ \dot{v}(t) = \frac{1}{m} \left[ -\alpha v(t) + Eu(t) + Gw(t) \right] \]

The equations for the controller are

\[ \dot{z}(t) = v_{\text{des}}(t) - v(t) \]
\[ u(t) = K_I \dot{z}(t) \]

The variable \( z \) is called the “integrator” in the control system, since it is a running integral of the error \( v_{\text{des}} - v \). Recall from circuit theory that an integrating circuit can be built with a single op-amp and capacitor (we will review this standard circuitry later).

To understand the closed-loop behavior, we need to eliminate \( u \) and \( z \) from the equations. This is done as follows - differentiate the car equation model, yielding

\[ \ddot{v}(t) = \frac{1}{m} \left[ -\alpha \dot{v}(t) + E\dot{u}(t) + G\dot{w}(t) \right] \]

Now \( \dot{u} \) appears, but this is known from the controller equations. In fact

\[ \dot{u}(t) = K_I \dot{z}(t) = K_I [v_{\text{des}}(t) - v(t)] \]

Substitute this into (34) to yield

\[ \ddot{v}(t) = \frac{1}{m} \left[ \alpha \dot{v}(t) + E K_I (v_{\text{des}}(t) - v(t)) - \alpha \dot{v}(t) + G\dot{w}(t) \right] \]

which can be manipulated into the form

\[ \ddot{v}(t) + \frac{\alpha}{m} \dot{v}(t) + \frac{E K_I}{m} v(t) = \frac{E K_I}{m} v_{\text{des}}(t) - \frac{G}{m} \dot{w}(t) \] (35)
In this final equation, \( w \) does not explicitly enter, only \( \dot{w} \) does. Hence, if \( w(t) \) approaches a constant limit, say \( \bar{w} \), then \( \dot{w} \to 0 \), and the particular final value of \( w \) (ie., \( \bar{w} \)) has no effect on the limiting value of \( v(t) \).

This is the amazing thing about integral control – it eliminates the steady-state error due to steady disturbances.

But, let’s go further, and try this out for a “realistic” car model, with \( m = 1000 \text{kg} \), \( E = 40 \), \( \alpha = 60 \), \( G = 98 \). The simulation is carried under the following conditions

- The initial speed of car is \( v(t_0) = 25 \text{ m/s} \).
- Initially, the desired speed is \( v_{\text{des}}(t) = 25 \text{ m/s} \).
- Initially, the hill inclination is \( w(t) = 0 \).
- The initial condition on control system integrator is \( z(t_0) = \frac{\alpha v_{\text{des}}}{EK} \).
- \( v_{\text{des}}(t) = 30 \text{ m/s} \) for \( t \geq 10 \text{s} \).
- \( w(t) = 3 \) for \( t \geq 160 \text{s} \).

Hence, at \( t = t_0 \), the value of \( u(t) \) is just right to maintain speed at \( 25 \text{ m/s} \), if there is no hill (check that at \( t = t_0 \), the equations for \( \dot{v} \) and \( \dot{z} \) give \( \dot{v}(t_0) = 0, \dot{z}(t_0) = 0 \)).

The trajectories for \( v_{\text{des}}(t) \) and \( w(t) \) for \( t > t_0 \) are shown below.
Four different plots of $v(t)$ are shown, for $K_I$ taking on values from 0.03 to 0.24. Although its not pretty, at steady-state, the value of $v(t)$ is equal to $v_{\text{des}}$, regardless of the hill. However, the response is generally unacceptable - either too oscillatory, or too sluggish.

Why is the response so sluggish and/or oscillatory? We need to study differential equations more carefully, as these types of responses were not possible with 1st order systems.

5.7 Problems

1. Work out the integral in the last line of equation (29), deriving equation (30).

2. Use the integrating factor method to derive the solution given in equation (27) to the differential equation (26).

3. A first-order system (input $u$, output $y$) has differential equation model

   $$\dot{y}(t) = ay(t) + bu(t)$$

   where $a$ and $b$ are some fixed numbers.

   (a) If the time constant is $\tau > 0$, and the steady-state gain is $\gamma$, solve for $a$ and $b$ in terms of $\tau$ and $\gamma$, Also solve for $\tau$ and $\gamma$ in terms of $a$ and $b$.

   (b) Suppose $\tau = 1$, $\gamma = 2$. Give the initial condition $y(0) = 4$, and the input signal $u$

   \[
   \begin{align*}
   u(t) &= 1 \quad \text{for } 0 \leq t < 5 \\
   u(t) &= 2 \quad \text{for } 5 \leq t < 10 \\
   u(t) &= 6 \quad \text{for } 10 \leq t < 10.1 \\
   u(t) &= 3 \quad \text{for } 10.1 \leq t < \infty.
   \end{align*}
   \]

   Sketch a reasonably accurate graph of $y(t)$ for $t$ ranging from 0 to 20. The sketch should be based on your understanding of a first-order system’s response, not by doing any particular integration.

   (c) Now suppose $\tau = 0.001$, $\gamma = 2$. Give the initial condition $y(0) = 4$, and the input signal $u$

   \[
   \begin{align*}
   u(t) &= 1 \quad \text{for } 0 \leq t < 0.005 \\
   u(t) &= 2 \quad \text{for } 0.005 \leq t < 0.01 \\
   u(t) &= 6 \quad \text{for } 0.01 \leq t < 0.0101 \\
   u(t) &= 3 \quad \text{for } 0.0101 \leq t < \infty.
   \end{align*}
   \]

   Sketch a reasonably accurate graph of $y(t)$ for $t$ ranging from 0 to 0.02. The sketch should be based on your understanding of a first-order system’s response, not by doing any particular integration. In what manner is this the “same” as part b?
4. Consider the car model, equation (25). The model parameters $m, \alpha, E$ are positive, while $G$ is negative. Suppose that for physical reasons, it is known that

$$u(t) \leq u_{\text{max}}$$

where $u_{\text{max}}$ is some fixed, upper bound on the available control action. For the questions below, assume that $w \equiv 0$ – we are on a flat road.

(a) In terms of the parameters, what is the maximum acceleration achievable by the car? Note: this will be a function of the current velocity.

(b) Starting from $v(0) = 0$, what is the minimum time, $T_{\text{min}}$, for the car to reach a speed $\bar{v}$. Your answer should be $T_{\text{min}}$ as a function of $\bar{v}$, and will depend on the model parameters.

(c) Take the parameters $\alpha = 60, E = 40, m = 1000$, where all are SI units. Also suppose that $u_{\text{max}} = 90$. Recompute your answer in part (a), and plot $T_{\text{min}}(\bar{v})$ versus $\bar{v}$ for $\bar{v}$ in the interval $[0, 30 \text{ meters/second}]$. Are the answers reasonable, and similar to cars that we drive?

(d) Qualitatively, explain how would things change if $F_{\text{drag}}$ had the form

$$F_{\text{drag}}(t) = -\alpha_1 v(t) - \alpha_2 v^2(t)$$

5. For the first-order linear system (constant coefficients $a$, $b$, and $c$)

$$\dot{x}(t) = ax(t) + bu(t)$$

$$y(t) = cx(t)$$

find the output $y(t)$ for $t \geq 0$ starting from the initial condition $x(0) = 0$, subject to input

- $u(t) = 1$ for $t \geq 0$
- $u(t) = \sin(\omega t)$ for $t \geq 0$

Plot $y(t)$ versus $t$ for the following cases:

(a) $a = -1, b = 1, c = 1, x(0) = 0, u(t) = 1$ for $t \geq 0$

(b) $a = -10, b = 10, c = 1, x(0) = 0, u(t) = 1$ for $t \geq 0$

(c) $a = -1, b = 1, c = 1, x(0) = 0, u(t) = \sin(0.1 \times t)$ for $t \geq 0$

(d) $a = -10, b = 10, c = 1, x(0) = 0, u(t) = \sin(0.1 \times t)$ for $t \geq 0$

(e) $a = -1, b = 1, c = 1, x(0) = 0, u(t) = \sin(t)$ for $t \geq 0$

(f) $a = -10, b = 10, c = 1, x(0) = 0, u(t) = \sin(t)$ for $t \geq 0$

(g) $a = -1, b = 1, c = 1, x(0) = 0, u(t) = \sin(10 \times t)$ for $t \geq 0$

(h) $a = -10, b = 10, c = 1, x(0) = 0, u(t) = \sin(10 \times t)$ for $t \geq 0$
Put cases (a), (b) on the same graph, cases (c), (d) on the same graph, cases (e), (f) on the same graph and cases (g), (h) on the same graph. Also, on each graph, also plot $u(t)$. Pick duration so that the limiting behavior is clear, but not so large that the graph is cluttered. Be sure and get the steady-state magnitude and phasing of the response $y$ (relative to $u$) correct.

6. With regards to your answers in problem 5,
   
   (a) Comment on the effect parameters $a$ and $b$ have on the step responses in cases (a)-(b).
   
   (b) Comment on the amplification (or attenuation) of sinusoidal inputs, and its relation to the frequency $\omega$.
   
   (c) Based on the speed of the response in (a)-(b) (note the degree to which $y$ “follows” $u$, even though $u$ has an abrupt change), are the sinusoidal responses in (c)-(h) consistent?

7. Consider the first-order linear system (constant coefficients $A$, $B$, $C$ and $D$)

$$
\dot{x}(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t) + Du(t)
$$

and assume that $A < 0$ (so the system is stable), $B = -A$, $C = 1$ and $D = 0$. Let $\alpha := -A$, so that the equations become

$$
\dot{x}(t) = \alpha [u(t) - x(t)] \\
y(t) = x(t)
$$

Let the initial condition be $x(0) = x_0$. Let $\omega \in \mathbb{R}$ be a given, fixed number. Suppose that the input $u$ is a sinusoid, for $t \geq 0$ $u(t) := \sin \omega t$.

   (a) (Trigonometry recap) Sketch a plot of $\tan(x)$ versus $x$, for all $x \in \mathbb{R}$. Also, sketch the inverse function $\tan^{-1}(w)$ versus $w$ for all $w$. Note that the inverse function is multivalued, since for any angle $\theta$ and any integer $n$, $\tan(\theta) = \tan(\theta + n\pi)$. From now on, when we refer to $\tan^{-1}$, we mean the function that for every $y \in \mathbb{R}$, $-\frac{\pi}{2} < \tan^{-1}(y) < \frac{\pi}{2}$, and $\tan(\tan^{-1}(y)) = y$.

   (b) Find expressions $M(\omega)$ and $\phi(\omega)$ such that

$$
x_{\omega,ss}(t) = M(\omega) \sin(\omega t + \phi(\omega)),
$$

where $A(\omega) \geq 0$ for all $\omega$. The angle $\phi$ will be known implicitly – you will be able to exactly describe what $\sin \phi$ and what $\cos \phi$ are. If you divide to write it in $\tan^{-1}$ form, make sure that you verify the rule we set out above.

   (c) Numerically fill in the table below
(d) Pick several values for $\alpha$. Use MatLab to graph $M(\omega)$ vs. $\omega$ on a log-log plot (log $\omega$ on the horizontal axis). In your graphs, choose $\omega_{\text{min}} \approx \frac{1}{1000} |A|$, choose $\omega_{\text{max}} \approx 1000 |A|$. Useful functions are `logspace`, `loglog` and `/` (period, slash). On each graph, hand-draw in “straightline” approximations which come from the entries in the table you derived above. Specifically, plot the 7 points, and connect them with straight lines. Qualitatively, compare the straightline approximation to the actual curves.

(e) Qualitatively explain how the function $M(\cdot)$ changes with parameter $\alpha$.

(f) Using the same values for $\alpha$, graph $\phi(\omega)$ vs. $\omega$ on a linear($\phi$)-log($\omega$) plot. Qualitatively explain how the function $\phi(\cdot)$ changes with $\alpha$. On each graph, hand-draw in “straightline” approximations which come from the entries in the table you derived above. Qualitatively, compare the straightline approximation to the actual curves.

(g) So, consider the cascade connection of two, first-order, stable, systems

\[
\begin{align*}
\dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t) \\
y_1(t) &= C_1 x_1(t) + D_1 u(t) \\
\dot{x}_2(t) &= A_2 x_2(t) + B_2 y_1(t) \\
y(t) &= C_2 x_2(t) + D_2 y_1(t)
\end{align*}
\]

By stable, we mean both $A_1 < 0$ and $A_2 < 0$. The cascade connection is shown pictorially below.

Suppose that the frequency response of System 1 is $M_1(\omega)$, $\phi_1(\omega)$ (or just the complex $G_1(\omega)$), and the frequency response of System 2 is $M_2(\omega)$, $\phi_2(\omega)$ (ie., the complex $G_2(\omega)$). Now, suppose that $\omega$ is a fixed real number, and $u(t) = \sin \omega t$. Show that the steady-state behavior of $y(t)$ is simply

\[
y_{ss}(t) = [M_1(\omega)M_2(\omega)]\sin (\omega t + \phi_1(\omega) + \phi_2(\omega))
\]

(h) Let $G$ denote the complex function representing the frequency response (forcing-frequency-dependent amplitude magnification $A$ and phase shift $\phi$, combined into
a complex number) of the cascaded system. How is \( G \) related to \( G_1 \) and \( G_2 \)?

**Hint:** Remember that for complex numbers \( G \) and \( H \),

\[
|GH| = |G||H|, \quad \angle (GH) = \angle G + \angle H
\]

8. The closed-loop cruise control system equations are given in equation (33). Modify the equations of the controller to include a measurement noise term, so

\[
u(t) = K_{ff}v_{des}(t) - K_{fb}\left(v(t) + n(t)\right)
\]

Here, \( n \) represents the measurement noise. The car parameters are given in Section 5.5.3.

(a) Define “outputs of interest” \( y_1 := v \) and \( y_2 := u \) (the actual speed and throttle).

Find \( A, B_1, B_2, C_1, \ldots, D_{23} \) so that the equations governing the closed-loop system are

\[
\begin{align*}
v(t) &= Av(t) + B_1v_{des}(t) + B_2w(t) + B_3n(t) \\
(y =)y_1(t) &= C_1v(t) + D_{11}v_{des}(t) + D_{12}w(t) + D_{13}n(t) \\
(u =)y_2(t) &= C_2v(t) + D_{21}v_{des}(t) + D_{22}w(t) + D_{23}n(t)
\end{align*}
\]

Your answer for the coefficients will be in terms of the car parameters, and the feedback gains. Assume that \( m, E \) and \( \alpha \) are positive.

(b) Under what conditions is the closed-loop system stable?

(c) What is the steady-state \((\omega = 0)\) gain from \( v_{des} \) to \( v \)? Your answer should be in terms of the car parameters, and \( K_{ff} \) and \( K_{fb} \).

(d) What is the steady-state \((\omega = 0)\) gain from \( w \) to \( v \)? Your answer should be in terms of the car parameters, and \( K_{ff} \) and \( K_{fb} \).

(e) What is the steady-state \((\omega = 0)\) gain from \( n \) to \( v \)? Your answer should be in terms of the car parameters, and \( K_{ff} \) and \( K_{fb} \).

(f) If \( K_{fb} \) is given, what is the correct value of \( K_{ff} \) so that the steady-state gain \((\omega = 0)\) from from \( v_{des} \) to \( v \) is exactly 1? Under that constraint, what are the values of the steady-state gain from \( w \) to \( v \) and \( n \) to \( v \)? What is the motivation for picking \( K_{fb} \) “large”? What is a drawback for picking \( K_{fb} \) large?

(g) Consider three cases, \( K_{fb} = 2, 4, 8 \). For each case, compute the appropriate value of \( K_{ff} \) based on steady-state gain considerations in part 8f above.

(h) Using Matlab (``subplot, loglog, abs, semilogx, angle``) make a 2-by-3 plot of the frequency response functions (ie., the sinusoidal steady-state gain) from the inputs \((v_{des}, w \) and \( n)\) to the outputs \((v \) and \( u)\). The plots should be layed out as shown below.
For magnitude plots, use \texttt{loglog}. For angle plots, use \texttt{semilogx}. Each axes should have 3 plots (associated with $K_{fb} = 2, 4, 8$ and the corresponding values for $K_{ff}$.

(i) Using Simulink, compute and plot the responses $v$ and $u$ for the system subjected to the inputs $v_{des}$ and $w$ given on page 38/39. Use $n(t) = 0$ for all $t$. Do this for the 3 different gain combinations. Plot the two “outputs of interest” in a 2-by-1 array of subplots, with $y_1$ in the top subplot, and $y_2$ in the bottom. Both axes should have 3 graphs in them.

(j) Certain aspects of the frequency-response curves are unaffected by the specific ($K_{fb}, K_{ff}$) values (eg., the graphs are the same regardless of the value, the graphs converge together in certain frequency ranges, etc). Mark these situations on your graphs, and in each case, explain why this is happening. Compare/connect to the time-response plots if possible/necessary.

(k) Certain aspects of the frequency-response curves are significantly affected by the specific ($K_{fb}, K_{ff}$) values (eg., the graphs have different magnitudes at low frequency, the graphs bend down (or up) at different frequencies, etc). Mark these situations on your graphs, and in each case, explain why this is happening. Compare/connect to the time-response plots if possible/necessary.

(l) At high frequency, the sinusoidal steady-state gain from $v_{des}$ to $u$ is not zero. This is easily observed in the frequency response plots. Where does this “show up” in the time response plots? Can you quantitatively connect the two plots?

(m) Notice how the frequency-response from $v_{des}$ to $v$ is nearly 1 (as a complex number) over a frequency range beginning at 0 and extending to some mid-frequency value. Beyond that value, the frequency response plot fades unremarkably to 0. Moreover, as we increase $K_{fb}$ the range of frequencies over which the frequency-response from $v_{des}$ to $v$ is nearly 1 is extended. Where does this “show up” in the time response plots? Can you quantitatively connect the two plots?

(n) Notice that as $K_{fb}$ increases, the magnitude of the frequency response from $w$ to $v$ decreases (less effect at high frequency though, and no effect at “infinite”
frequency). Where does this “show up” in the time response plots? Can you quantitatively connect the two plots?

(o) Using Simulink, compute and plot the responses \( v \) and \( u \) for the system subjected to the inputs \( v_{\text{des}} \) and \( w \) given on page 36. From the Sources library, use the Band-Limited White Noise block. Set the noise power to about 0.033, and the sample time to 0.1 seconds. Set the Seed field to \( \text{clock}*[0;0;0;0;0;1] \). This means that every time you simulate, Matlab will call the clock function, which returns a 1-by-6 array (year,month,day,hour,minute,second) and then “pick off” the “seconds” value. This is used as the seed in the random number generator, ensuring that you get a different noise trajectory each time you simulate. Do this for the 3 different gain combinations. Plot the two “outputs of interest” in a 2-by-1 array of subplots, with \( y_1 \) in the top subplot, and \( y_2 \) in the bottom. Both axes should have 3 graphs in them. Comment on the effect that measurement noise has on the outputs, and how the increased feedback gain affects this.

(p) In doing the simulation, we chose (nearly) a constant for \( w \), but a quickly, randomly varying signal for \( n \)? Why? What is the interpretation of these two signals?

(q) Look at the frequency response plots governing the effect that \( n \) has on \( v \) and \( u \). Are these consistent with what you see in part 8o?

One aspect to understanding the performance of closed-loop systems is getting comfortable with the relations between frequency responses and time responses.

9. Re-read “Leibnitz’s” rule in your calculus book, and consider the time-varying differential equation

\[
\dot{x}(t) = a(t)x(t) + b(t)u(t)
\]

with \( x(0) = x_0 \). Show, by substitution, or integrating factor, that the solution to this is

\[
x(t) = e^{\int_0^t a(\xi)d\xi}x_o + \int_0^t e^{\int_\xi^t a(\eta)d\eta}b(\tau)u(\tau)d\tau
\]

10. Suppose two systems are interconnected, with individual equations given as

\[
S_1: \quad \dot{y}(t) = [u(t) - y(t)] \\
S_2: \quad u(t) = 2[y(t) - r(t)]
\]

(a) Consider first \( S_1 \) (input \( u \), output \( y \)): Show that for any initial condition \( y_0 \), if \( u(t) \equiv \bar{u} \) (a constant), then \( y(t) \) approaches a constant \( \bar{y} \), that only depends on the value of \( \bar{u} \). What is the steady-state gain of \( S_1 \)?

(b) Next consider \( S_2 \) (input \( (r, y) \), output \( u \)): Show that if \( r(t) \equiv \bar{r} \) and \( y(t) \equiv \bar{y} \) (constants), then \( u(t) \) approaches a constant \( \bar{u} \), that only depends on the values \( (\bar{r}, \bar{y}) \).

(c) Now, assume that the closed-loop system also has the steady-state behavior – that is, if \( r(t) \equiv \bar{r} \), then both \( u(t) \) and \( y(t) \) will approach limiting values, \( \bar{u} \) and \( \bar{y} \), only dependent on \( \bar{r} \). Draw a block-diagram showing how the limiting values are related, and solve for \( \bar{u} \) and \( \bar{y} \) in terms of \( \bar{r} \).
(d) Now check your answer in part 10c. Suppose \( y(0) = 0 \), and \( r(t) = 1 =: \bar{r} \) for all \( t \geq 0 \). Eliminate \( u \) from the equations 36, and determine \( y(t) \) for all \( t \). Make a simple graph. Does the result agree with your answer in part 10c?

Lesson: since the assumption we made in part 10c was actually not valid, the analysis in part 10c is incorrect. That is why, for a closed-loop steady-state analysis to be based on the separate component’s steady-state properties, we must know from other means that the closed-loop system also has steady-state behavior.

11. Suppose two systems are interconnected, with individual equations given as

\[
\begin{align*}
S_1: \quad \dot{y}(t) &= [u(t) + y(t)] \\
S_2: \quad u(t) &= 2 [r(t) - y(t)]
\end{align*}
\] (37)

(a) Consider first \( S_1 \) (input \( u \), output \( y \)): If \( u(t) \equiv \bar{u} \) (a constant), then does \( y(t) \) approach a constant \( \bar{y} \), dependent only on the value of \( \bar{u} \)?

(b) Next consider \( S_2 \) (input \( (r, y) \), output \( u \)): If \( r(t) \equiv \bar{r} \) and \( y(t) \equiv \bar{y} \) (constants), then does \( u(t) \) approach a constant \( \bar{u} \), dependent only on the values \( \bar{r}, \bar{y} \)?

(c) Suppose \( y(0) = y_0 \) is given, and \( r(t) =: \bar{r} \) for all \( t \geq 0 \). Eliminate \( u \) from the equations 37, and determine \( y(t) \) for all \( t \). Also, plugging back in, determine \( u(t) \) for all \( t \). Show that \( y \) and \( u \) both have limiting values that only depend on the value \( \bar{r} \), and determine the simple relationship between \( \bar{r} \) and \((\bar{y}, \bar{u})\).

Lesson: Even though \( S_1 \) does not have steady-state behavior on its own, in feedback with \( S_2 \), the overall closed-loop system does.
12. Consider a first-order system $P$, with inputs $(d, u)$ and output $y$, governed by

$$\begin{align*}
\dot{x}(t) &= ax(t) + b_1 d(t) + b_2 u(t) \\
y(t) &= cx(t)
\end{align*}$$

(a) Assume $P$ is stable (i.e., $a < 0$). For $P$ itself, what is the steady-state gain from $u$ to $y$ (assuming $d \equiv 0$)? Call this gain $G$. What is the steady-state gain from $d$ to $y$ (assuming $u \equiv 0$)? Call this gain $H$.

(b) $P$ is controlled by a “proportional” controller of the form

$$u(t) = K_1 r(t) + K_2 [r(t) - (y(t) + n(t))]$$

Here, $r$ is the reference signal (the desired value of $y$), $n$ is the measurement noise (so that $y + n$ is the measurement of $y$), $K_1$ and $K_2$ are gains to be chosen. By substituting for $u$, write the differential equation for $x$ in the form

$$\begin{align*}
\dot{x}(t) &= Ax(t) + B_1 r(t) + B_2 d(t) + B_3 n(t)
\end{align*}$$

Also, express the output $y$ and control input $u$ as functions of $x$ and the external inputs $(r, d, n)$ as

$$\begin{align*}
y(t) &= C_1 x(t) + D_{11} r(t) + D_{12} d(t) + D_{13} n(t) \\
u(t) &= C_2 x(t) + D_{21} r(t) + D_{22} d(t) + D_{23} n(t)
\end{align*}$$

All of the symbols $(A, B_1, \ldots, D_{23})$ will be functions of the lower-case given symbols and the controller gains. Below, we will “design” $K_1$ and $K_2$ two different ways, and assess the performance of the overall system.

(c) Under what conditions is the closed-loop system is stable? What is the steady-state gain from $r$ to $y$ (assuming $d \equiv 0$ and $n \equiv 0$)? What is the steady-state gain from $d$ to $y$ (assuming $r \equiv 0$ and $n \equiv 0$)?

(d) Design a “feed-forward” controller based on the steady-state gain $G$. In particular, pick $K_1 := \frac{1}{G}$, and $K_2 := 0$. Note that we are now considering a feedback control system that actually had no feedback ($K_2 = 0$). Nevertheless, I refer to the system as the “closed-loop” system.

i. For the “closed-loop” system, compute the steady-state gains from all external inputs $(r, d, n)$ to the two “outputs” $(y, u)$.

ii. Comment on the steady-state gain from $r \rightarrow y$.

iii. What is the sensitivity of the steady-state gain from $r \rightarrow y$ to the parameter $b_2$? Here you should treat $K_1$ as a fixed number.

iv. Comment on the relationship between the open-loop steady-state gain from $d \rightarrow y$ (i.e., $H$ computed above) and the closed-loop computed in part 12(d)i.

v. Comment on the steady-state gain from $d$ to $u$. Based on $d$’s eventual effect on $u$, is the answer in part 12(d)iv surprising?

vi. Comment on the steady-state gain from $n$ to both $y$ and $u$. 

vii. What is the time-constant of the closed-loop system.

viii. In this part we have considered a feedback control system that actually had no feedback ($K_2 = 0$). So the closed-loop system is in fact an open-loop system. Summarize the effect of the open-loop control strategy (called “feedforward,” since the control input is actually just a function of the reference signal $r$, “fed-forward to the process”). In this summary, include a comparison of the process time constant, and the closed-loop time constant, as well as the tracking capabilities (how $y$ follows $r$), the sensitivity of the tracking capabilities to parameter changes, and the disturbance rejection properties.

(e) Now design a true feedback control system. Pick $K_2$ so that the closed-loop steady-state gain from $d \rightarrow y$ is at least 5 times less than the uncontrolled steady-state gain from $d \rightarrow y$ (which we called $H$). Constrain your choice of $K_2$ so that the closed-loop system is stable. Since we are working fairly general, for simplicity, you may assume $a < 0$ and $b_1 > 0, b_2 > 0$ and $c > 0$.

i. With $K_2$ chosen, pick $K_1$ so that the closed-loop steady-state gain from $r \rightarrow y$ is 1.

ii. With $K_1$ and $K_2$ both chosen as above, what is the sensitivity of the steady-state gain from $r \rightarrow y$ to the parameter $b_2$?

iii. What is the time-constant of the closed-loop system?

iv. What is the steady-state gain from $d \rightarrow u$? How does this compare to the previous case (feedforward)?

v. With $K_2 \neq 0$, does the noise $n$ now affect $y$?

(f) Let’s use specific numbers: $a = -1, b_1 = 1, b_2 = 1, c = 1$. Summarize all computations above in a table – one table for the feedforward case, and one table for the true feedback case. Include in the table all steady-state gains, time constant, and sensitivity of $r \rightarrow y$ to $b_2$.

(g) Plot the frequency responses from all external inputs to both outputs. Do this in a $2 \times 3$ matrix of plots that I delineate in class. Use Matlab, and the subplot command. Use a frequency range of $0.01 \leq \omega \leq 100$. There should be two lines on each graph.

(h) Indicate on the graph how the true feedback system accomplishes tracking, disturbance rejection, and lower time-constant, but increased sensitivity to noise.

(i) Keeping $K_1$ and $K_2$ fixed, change $b_2$ from 1 to 0.8. Redraw the frequency responses, now including all 4 lines. Indicate on the graph how the true feedback system accomplishes good tracking that is more insensitive to process parameter changes than the feedforward system.