9 More on ODEs

Recall the equations from the cruise control system

\[ \begin{align*}
\dot{v}(t) &= -\frac{\alpha}{m} v(t) + \frac{E}{m} u(t) + \frac{G}{m} w(t) \\
\dot{z}(t) &= v_{\text{des}}(t) - v(t) \\
v_{\text{meas}}(t) &= v(t) + n(t) \\
u(t) &= K_I z(t) + K_f f v_{\text{des}}(t) - K_f b v(t)
\end{align*} \]  

(49)

Note that these are well defined, whether or not \( w; n \) and \( v_{\text{des}} \) are differentiable functions. As two coupled 1st order differential equations, (49) is in the appropriate form for a numerical differential equation solver, such as Simulink. Unfortunately, based on what knowledge we have now, the characteristic equation is not evident, and for that we need to eliminate variables, and obtain one 2nd order differential equation. We assume all of the inputs \( w; n \) and \( v_{\text{des}} \) are differentiable, and we differentiate the first equation, substituting for \( \dot{u} \), yielding

\[ \dot{v}(t) + \frac{\alpha}{m} \dot{v}(t) + \frac{E K_f b}{m} \dot{v}(t) + \frac{E K_I}{m} v(t) = \frac{E K_f f}{m} v_{\text{des}}(t) + \frac{G}{m} \dot{w}(t) - \frac{E K_I}{m} n(t) - \frac{E K_f b}{m} \dot{n}(t) \]

(50)

**Question:** It appears that \( w; n \) and \( v_{\text{des}} \) must be differentiable for this relationship to hold. That would mean that this is not suitable for determining the response to step changes in \( w \) and/or \( v_{\text{des}} \). In this and the next section, we learn about *distributions*, and how to make sense of differential equations when the right hand side is not classically defined.

Another issue arises when we simulate this system with gains set to appropriate values. Suppose that with the usual values for \( E, \alpha, m \) and \( G \), the controller gains \( K_f b \) and \( K_I \) are chosen so that \( \xi = 0.707, \omega_n = 0.7 \). Disregard \( w \) and \( n \), leaving

\[ \dot{v}(t) + \frac{\alpha}{m} \dot{v}(t) + \frac{E K_f b}{m} \dot{v}(t) + \frac{E K_I}{m} v(t) = \frac{E K_f f}{m} v_{\text{des}}(t) + \frac{E K_f f}{m} \dot{v}_{\text{des}}(t) \]

(51)

Further, suppose we chose \( K_f f = \{0 \ast K_f b, 0.5 K_f b, K_f b, 2 K_f b\} \). Step-responses \( (v_{\text{des}} \text{ starts at 0, and abruptly changes to 1}) \) with all initial conditions \( (v \text{ and } \dot{v} \text{ set to 0}) \) for these cases are shown. Note that the overshoot is a function of \( K_f f \), even though \( K_f f \) has no effect on the roots of the characteristic equation.
This difference must be due to the presence of the $v_{\text{des}}$ term on the right-hand side of the differential equation. This section will uncover exactly how this term affects the response.

### 9.1 Introduction

Suppose that through manipulation, we derive a relationship between the input ($u$) and output ($y$) of a system to be

$$\ddot{y}(t) + a_1 \dot{y}(t) + a_2 y(t) = b_1 \dot{u}(t) + b_2 u(t)$$

Next, suppose that we are given initial conditions $y_0$ and $\dot{y}_0$, and are asked to compute the response to a unit-step input, applied just after the system is released from the initial conditions. Note that the input is not differentiable at $t = 0$, and hence the right-hand side of the differential equation is not well defined. How do we get a solution in that case? In order to study this, we first consider particular solutions to a special class of nonconstant forcing functions.

### 9.2 Other Particular Solutions

We have already seen that the differential equation

$$y^{[n]}(t) + a_1 y^{[n-1]}(t) + \cdots + a_{n-1} \dot{y}(t) + a_n y(t) = 1$$

has a particular solution $y_p(t) = \frac{1}{a_n}$ for all $t$. Recall that all particular solutions can be generated by adding all homogeneous solutions to any one particular solution.
What about a particular solution to the equation

\[ y^{[n]}(t) + a_1 y^{[n-1]}(t) + \cdots + a_{n-1} y(t) + a_n y(t) = \gamma_0 + \gamma_1 t \]

where \( \gamma_0 \) and \( \gamma_1 \) are some fixed constants. If we differentiate both sides, we see that a particular solution is \( \dot{y} \) taking a constant value, and hence we “guess” that there is a particular solution of the form

\[ y_p(t) = \alpha t + \beta \]

for some proper choice of constants \( \alpha \) and \( \beta \). Indeed, note that \( \dot{y}_p = \alpha \), and all higher order derivatives are identically zero. Plugging in, and equating gives

\[ a_{n-1} \alpha + a_n (\alpha t + \beta) = \gamma_0 + \gamma_1 t \]

which must hold for all \( t \). This gives

\[ a_{n-1} \alpha + a_n \beta = \gamma_0, \quad a_n \alpha = \gamma_1 \]

which can be easily solved, yielding

\[ \alpha = \frac{\gamma_1}{a_n}, \quad \beta = \frac{\gamma_0 - a_{n-1} \alpha}{a_n} \]

This approach is easy to generalize for forcing functions of the form \( \sum_{k=0}^{n} \gamma k t^k \).

### 9.3 Limits approaching steps

Return to the original problem, which is computing the step-response of

\[ \ddot{y}(t) + a_1 \dot{y}(t) + a_2 y(t) = b_1 \dot{u}(t) + b_2 u(t) \]

subject to initial conditions \( y(0) = Y_0, \dot{y}(0) = \dot{Y}_0 \), and the input \( u(t) = \mu(t) \).

We begin with a simpler problem, which avoids the differentiability problems of the unit step function. For each \( \epsilon > 0 \), define a function \( \mu_\epsilon \) as

\[ \mu_\epsilon(t) := \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{\epsilon} t & \text{for } 0 \leq t < \epsilon \\ 1 & \text{for } \epsilon \leq t \end{cases} \]

Note that this is continuous, and piecewise differentiable, with derivative

\[ \dot{\mu}_\epsilon(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{\epsilon} & \text{for } 0 < t < \epsilon \\ 0 & \text{for } \epsilon < t \end{cases} \]

We want to determine the solution of the differential equation (52) subject to initial conditions \( y(0) = Y_0, \dot{y}(0) = \dot{Y}_0 \), and the input \( u(t) = \mu_\epsilon(t) \). Let the solution be denoted by \( y_\epsilon(t) \).

We break the problem into 2 time intervals: \((0, \epsilon)\) and \((\epsilon, \infty)\).
• Over the first interval, \((0, \epsilon)\), we explicitly compute the solution in the standard manner, finding a particular solution, and then adding in the appropriate homogeneous solution to match up the initial conditions at 0.

• Over the second interval, we follow the same procedure (since the qualitative behavior of the right-hand-side of the differential equation is different in this time interval, we need to start with a new particular solution), last values (ie., the values at \(t = \epsilon\)) of the solution obtained in the first interval as initial conditions.

In order to achieve our original objective (response due to unit-step), after completing the solution in the first interval, we take the limit as \(\epsilon \to 0\) in order to compute the “initial conditions” for the second interval.

Reiterating – we want to determine properties of the solution at \(t = \epsilon\), in particular we want to determine \(y_\epsilon(\epsilon)\), and \(\dot{y}_\epsilon(\epsilon)\), and then take the limit as \(\epsilon \to 0\) (ie., as the input function becomes more and more like a unit-step). We will denote these values as \(y(0^+)\) and \(\dot{y}(0^+)\); the values of \(y\) and \(\dot{y}\) just after the step change occurs. They become the initial conditions for the solution after the input’s step change has taken place.

During the interval \((0, \epsilon)\), the right-hand side of (52) is well defined, and is

\[
b_1 \dot{u}(t) + b_2 u(t) = \frac{b_2}{\epsilon} t + \frac{b_1}{\epsilon}
\]

A particular solution (using the method we derived in Section 9.2) is \(y_p(t) = \alpha t + \beta\), where

\[
\alpha = \frac{b_2}{a_2 \epsilon}, \quad \beta = \frac{b_1 a_2 - a_1 b_2}{a_2^2 \epsilon}
\]

Let \(\lambda_1, \lambda_2\) be the roots of the characteristic polynomial,

\[
\lambda^2 + a_1 \lambda + a_2 = 0
\]

For now, let’s assume that they are distinct (you could rework what we do here for the case when they are not distinct - it all works out the same). For a later calculation, please check that \(a_1 = -\lambda_1 - \lambda_2\), and \(a_2 = \lambda_1 \lambda_2\). Hence, the solution to the differential equation for \(t > 0\) must be

\[
y_\epsilon(t) = \alpha t + \beta + c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}
\]

where \(c_1\) and \(c_2\) are uniquely chosen to satisfy the initial conditions \(y_\epsilon(0) = Y_0, \dot{y}_\epsilon(0) = \dot{Y}_0\).

Differentiating gives that

\[
\dot{y}_\epsilon(t) = \alpha + c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t}
\]

Satisfying the initial conditions at \(t = 0\) gives conditions that the constants \(c_1\) and \(c_2\) must satisfy

\[
\begin{bmatrix}
1 & 1 \\
\lambda_1 & \lambda_2
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= \begin{bmatrix}
Y_0 - \beta \\
\dot{Y}_0 - \alpha
\end{bmatrix}
\]
which can be solved as

\[
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix}
\lambda_2 & -1 \\
-\lambda_1 & 1
\end{bmatrix} \begin{bmatrix} Y_0 - \beta \\ Y_0 - \alpha \end{bmatrix}
\]

In terms of \(c_1\) and \(c_2\), the value of \(y_e\) and \(\dot{y}_e\) at \(t = \epsilon\) are (using equations (53 and 54))

\[
\begin{bmatrix}
y_e(\epsilon) \\
\dot{y}_e(\epsilon)
\end{bmatrix} = \begin{bmatrix}
e^{\lambda_1 \epsilon} & e^{\lambda_2 \epsilon} \\
\lambda_1 e^{\lambda_1 \epsilon} & \lambda_2 e^{\lambda_2 \epsilon}
\end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \alpha \epsilon + \beta \\ \alpha \end{bmatrix}
\]

Substituting, gives

\[
\begin{bmatrix}
y_e(\epsilon) \\
\dot{y}_e(\epsilon)
\end{bmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix}
\lambda_2 e^{\lambda_1 \epsilon} - \lambda_1 e^{\lambda_2 \epsilon} & e^{\lambda_2 \epsilon} - e^{\lambda_1 \epsilon} \\
\lambda_2 \lambda_1 (e^{\lambda_1 \epsilon} - e^{\lambda_2 \epsilon}) & \lambda_2 e^{\lambda_2 \epsilon} - \lambda_1 e^{\lambda_1 \epsilon}
\end{bmatrix} \begin{bmatrix} Y_0 - \beta \\ Y_0 - \alpha \end{bmatrix} + \begin{bmatrix} \alpha \epsilon + \beta \\ \alpha \end{bmatrix}
\]

Rearranging gives

\[
\begin{bmatrix}
y_e(\epsilon) \\
\dot{y}_e(\epsilon)
\end{bmatrix} = \begin{bmatrix}
y_e(\epsilon) \\
\dot{y}_e(\epsilon)
\end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} \alpha \epsilon + \beta \\ \alpha \end{bmatrix}
\]

For notational purposes, let

\[
M_\epsilon := \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix}
\lambda_2 e^{\lambda_1 \epsilon} - \lambda_1 e^{\lambda_2 \epsilon} & e^{\lambda_2 \epsilon} - e^{\lambda_1 \epsilon} \\
\lambda_2 \lambda_1 (e^{\lambda_1 \epsilon} - e^{\lambda_2 \epsilon}) & \lambda_2 e^{\lambda_2 \epsilon} - \lambda_1 e^{\lambda_1 \epsilon}
\end{bmatrix}
\]

Then, we have

\[
\begin{bmatrix}
y_e(\epsilon) \\
\dot{y}_e(\epsilon)
\end{bmatrix} = M_\epsilon \begin{bmatrix} Y_0 - \beta \\ Y_0 - \alpha \end{bmatrix} + \begin{bmatrix} \alpha \epsilon + \beta \\ \alpha \end{bmatrix}
\]

This is further manipulated into

\[
\begin{bmatrix}
y_e(\epsilon) \\
\dot{y}_e(\epsilon)
\end{bmatrix} = M_\epsilon \begin{bmatrix} Y_0 \\ Y_0 \end{bmatrix} + \left( -M_\epsilon + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) \begin{bmatrix} \beta \\ \alpha \end{bmatrix}
\]

But recall, \(\alpha\) and \(\beta\) depend on \(\epsilon\), so that should be substituted,

\[
\begin{bmatrix}
y_e(\epsilon) \\
\dot{y}_e(\epsilon)
\end{bmatrix} = M_\epsilon \begin{bmatrix} Y_0 \\ Y_0 \end{bmatrix} + \frac{1}{\epsilon} \left( -M_\epsilon + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) \begin{bmatrix} b_0 a_2 - a_1 b_2 \\ b_2 \\ a_2 \end{bmatrix}
\]

Note that

\[
\lim_{\epsilon \to 0} M_\epsilon = I_2
\]

and using L’Hospital’s rule,

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( -M_\epsilon + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ \lambda_2 \lambda_1 - \lambda_2 - \lambda_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ a_2 \\ a_1 \end{bmatrix}
\]
Hence,

\[
\begin{bmatrix}
  y(0^+) \\
  \dot{y}(0^+)
\end{bmatrix} := \lim_{\epsilon \to 0, \epsilon > 0}
\begin{bmatrix}
  y_0(\epsilon) \\
  \dot{y}_0(\epsilon)
\end{bmatrix} = \begin{bmatrix}
  Y_0 \\
  \dot{Y}_0 + b_1
\end{bmatrix}
\]

In summary, suppose the input waveform is such that the right-hand side (i.e., forcing function) of the differential equation has singularities at certain discrete instants in time. Between those instants, all terms of the differential equation are well-behaved, and we solve it in the classical manner. However, at the instants of the singularity, some specific high order derivative of \( y \) experiences a discontinuity in its value, and the discontinuity can be determined from the differential equation. Essentially, we get “new” initial conditions for the differential equation just after the singularity in terms of the solution of the ODE just before the singularity and the ODE itself.

We have seen that the values of \( y \) and \( \dot{y} \) at a time just before the input singularity are related to the values of \( y \) and \( \dot{y} \) just after the singularity.

In the next section we learn some tricks that allow us to redo this type of calculation for general systems (not just second order). Distributions are mathematical constructions that, in a rigorous manner, allow us to repeat the rigorous calculation we just did in this section in a less tedious fashion.

Another point to notice: this calculation shows that the term \( b_1 \dot{u}(t) \), which appears on the right-hand side of the ODE in equation (52) plays a role in the solution. So, both the left and right sides of an input/output ODE have qualitative and quantitative effects of the solution. We will formalize this as we proceed...

### 9.4 Problems

1. Consider the differential equation

   \[
   \ddot{y}(t) + 4\dot{y}(t) + 3y(t) = b_1 \dot{u}(t) + 3u(t),
   \]

   subject to the forcing function \( u(t) := \mu(t) \), and initial conditions \( y(0^-) = 0, \dot{y}(0^-) = 0 \).

   (a) Determine:
   - the conditions of \( y \) (and \( \dot{y} \)) at \( t = 0^+ \),
   - the roots of the homogeneous equation (and their values of \( \xi \) and \( \omega_n \) if complex)
   - the final value, \( \lim_{t \to \infty} y(t) \).
   
   Some of these will be dependent on the parameter \( b_1 \).

   (b) Based on your answers, sketch a guess of the solution for 8 cases:

   \[
   b_1 = -6, -3, -0.3, -0.03, 0.03, 0.3, 3, 6.
   \]

   Sketch them on the same axes.
(c) Determine the exact solution (by hand). Do as much of this symbolically as you can (in terms of $b_1$) so that you only need to the majority of the work once, and then plug in the values of $b_1$ several times. Use the `plot` command in MatLab (or other) to get a clean plot of the solution $y(t)$ from $0^+$ to some suitable final time. Plot them on the same graph.