NOTE: Any unmarked summing junctions are positively signed (+).

Shown below is a graph of

\[
\frac{1}{\sqrt{1 - \xi^2}} e^{-\xi x} \sin \left( \sqrt{1 - \xi^2} x \right)
\]

versus \( x \), for 7 values of \( \xi \) evenly spaced between 0.3 and 0.9.
1. (a) What is the general form of real (as opposed to complex) solutions to the differential equation

\[ \ddot{x}(t) + 4\dot{x}(t) + 13x(t) = 0 \]

(b) What is the general form of real solutions to the differential equation

\[ \ddot{x}(t) + 4\dot{x}(t) + 13x(t) = -26 \]

Your expressions both should have two free constants. I would like there to be no \(\sqrt{-1}\) in the answers, just exponentials, \(\cos\) and \(\sin\). HINT: Although the roots are complex, and the \((\xi, \omega)\) parametrization certainly may be used, it will be “less messy” to compute the roots as complex numbers, and examine their real/imaginary parts.
2. Shown below are two systems. The system on the left is the **nominal system**, while the system on the right represents a deviation from the nominal (the insertion of the dashed box) and is called the **perturbed system**.

Based on the values of $K_P$ and $K_I$, and some analysis, you should have a general idea of how the nominal system behaves (e.g., the effect of $r$ on $u$ and $y$). Consider 3 different possibilities (listed below) regarding the relationship between the nominal and perturbed systems:

(a) The perturbed system behaves pretty much the same as the nominal system.
(b) The perturbed system behaves quite differently from the nominal system, but is still stable.
(c) The perturbed system is unstable.

For each row in the table below, which description from above applies? Write **a**, **b**, or **c** in each box. Show work below.

<table>
<thead>
<tr>
<th>$K_P$</th>
<th>$K_I$</th>
<th>$\beta$</th>
<th>Your Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.8</td>
<td>4</td>
<td>0.02</td>
<td></td>
</tr>
<tr>
<td>1.4</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>100</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>2500</td>
<td>0.02</td>
<td></td>
</tr>
</tbody>
</table>
3. Neatly/accurately sketch the solution (for $t \geq 0$) to the differential equation

$$\dot{x}(t) = -2x(t) + u(t)$$

subject to the initial condition $x(0) = 2$, and forcing function

$$u(t) = 0 \quad \text{for } 0 \leq t \leq 2.5$$
$$u(t) = -2 \quad \text{for } 2.5 < t \leq 4.5$$
$$u(t) = 8 \quad \text{for } 4.5 < t \leq 6$$
4. Suppose $0 < \xi < 1$ and $\omega_n > 0$. Consider the system

\[ \ddot{y}(t) + 2\xi \omega_n \dot{y}(t) + \omega_n^2 y(t) = \omega_n \dot{u}(t) \]

subject to the initial conditions $y(0^-) = 0, \dot{y}(0^-) = 0$, and the unit-step forcing function, namely $u(t) = 0$ for $t = 0$, and $u(t) = 1$ for $t > 0$. Show that the response is

\[ y(t) = \frac{1}{\sqrt{1 - \xi^2}} e^{-\xi \omega_n t} \sin \left( \sqrt{1 - \xi^2} \omega_n t \right) \]
5. A process, with input \( u \), disturbance \( d \) and output \( y \) is governed by

\[
\dot{y}(t) = y(t) + u(t) + d(t)
\]

(a) Is the process stable?

(b) Suppose \( y(0) = -3 \), and \( u(t) = d(t) \equiv 0 \) for all \( t \geq 0 \). What is the solution \( y(t) \) for \( t \geq 0 \).

(c) A PI (Proportional plus Integral) controller is proposed

\[
\begin{align*}
    u(t) &= K_P [r(t) - y(t)] + K_I z(t) \\
    \dot{z}(t) &= r(t) - y(t)
\end{align*}
\]

Eliminate \( z \) and \( u \), and determine the closed-loop differential equation relating the variables \( (y, r, d) \).
(d) For what values of $K_P$ and $K_I$ is the closed-loop system stable?

(e) The closed-loop system is 2nd order. What are the appropriate values of $K_P$ and $K_I$ so that the closed-loop system characteristic polynomial has roots described by $\xi = 0.8, \omega_n = 0.5$?

(f) What are the appropriate values of $K_P$ and $K_I$ so that the closed-loop system characteristic polynomial has roots described by $\xi = 0.8, \omega_n = 1.0$?

(g) What are the appropriate values of $K_P$ and $K_I$ so that the closed-loop system characteristic polynomial has roots described by $\xi = 0.8, \omega_n = 2.0$?
(h) For each of the 3 cases above, accurately sketch the response of $y$ due to a unit-step disturbance $d$, assuming $r$ is identically zero, and assuming all initial conditions are zero.

<table>
<thead>
<tr>
<th>Time, $t$</th>
<th>y(t) Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>0.4</td>
</tr>
<tr>
<td>1.5</td>
<td>0.6</td>
</tr>
<tr>
<td>2</td>
<td>0.8</td>
</tr>
<tr>
<td>2.5</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1.2</td>
</tr>
</tbody>
</table>
(i) Finally, take a different viewpoint – eliminate $z$ and $y$, and determine the closed-loop differential equation relating the variables $(u, r, d)$. The variables $r$ and $d$ will be the inputs, and $u$ will be the output (it is “caused” by $r$ and $d$).