ME 132, Spring 2004, Quiz 2

Name:

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NOTE: Any unmarked summing junctions are positively signed (+).
1. The transfer function of a controller is $C(s) = \frac{K_p s + K_i}{s}$, and the transfer function of a process is $G(s) = \frac{2}{\tau s + 1}$. A diagram of the closed-loop system is shown below.

(a) What is the closed-loop transfer function from $R$ to $Y$.

(b) What is the closed-loop transfer function from $R$ to $U$. 
(c) What is the closed-loop transfer function from $D$ to $Y$.

(d) What is the closed-loop transfer function from $D$ to $U$. 

(e) What is the differential equation relating $r$ and $d$ to $y$?

(f) Under what conditions (on $K_P, K_I, \tau, \gamma$) is the closed-loop system stable?
2. A block diagram is shown below. Each system is represented by its transfer function.

(a) In terms of the transfer functions of the individual blocks, what is the transfer function from \( R \) to \( Y \)?
(b) In terms of the transfer functions of the individual blocks, what is the transfer function from $D$ to $Y$?
3. A block diagram is shown below. Each system is represented by its transfer function.

(a) In terms of $K_P$, $K_I$, $K_D$, $\tau$, what is the transfer function from $R$ to $Y$ (hint: the denominator should be 4th order).
(b) In terms of $K_P, K_I, K_D, \tau$, what is the transfer function from $D$ to $Y$ (hint: same as above).

(c) In terms of $K_P, K_I, K_D, \tau$, what is the characteristic polynomial of the closed-loop system?

(d) In terms of $K_P, K_I, K_D, \tau$, what is the differential equation relating $r$ and $d$ to $y$?
4. In this problem, we carefully study the response of the first-order system described by

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

Here \( u \) is the input, and \( y \) is the output. All of the coefficients \( A, B, C \) and \( D \) are just real numbers. In this problem, we are going to emphasize the case when \( D \neq 0 \). To have things makes sense, we need to explicitly assume that \( C \neq 0 \).

(a) Show that the transfer function description is

\[
Y = \left[ D + \frac{CB}{s - A} \right] U
\]

(b) By dividing out constants from numerator and denominator, show that the transfer function can also be written as

\[
Y = \frac{(D)(\frac{1}{-A})s + (D + \frac{CB}{-A})}{(\frac{1}{-A})s + 1} U
\]
(c) Now, temporarily return to the differential equation description involving $x$, $u$ and $y$. If $u$ changes abruptly (i.e., discontinuous jump) by 1 unit, by how much does $y$ change abruptly (what is the size of the discontinuous jump in $y$)? Call this number the “instantaneous-gain” of the system.

(d) If $u(t)$ is a constant for $t \geq 0$, say $\bar{u}$, then

$$x(t) = \frac{CB}{-A} \bar{u} + e^{At} \left( x(0) - \frac{CB}{-A} \bar{u} \right)$$

Obviously, if $A < 0$, then during a time-interval when $u$ is constant, $x(t)$ transitions from its initial value towards its final value with an exponential decay characterized by the time-constant. This is all stuff we know. Translate this (using $y(t) = Cx(t) + Du(t)$) to show how $y$ evolves once $u$ is a constant: If $u(t)$ is a constant for $t \geq 0$, say $\bar{u}$, show that

$$y(t) = \left( D + \frac{CB}{-A} \right) \bar{u} + e^{At} \left[ y(0) - \left( D + \frac{CB}{-A} \right) \bar{u} \right].$$

Note that if $A < 0$, this implies (important fact to remember) that during a time-interval when $u$ is constant, $y(t)$ transitions from its initial value towards its final value with an exponential decay characterized by the time-constant.
(e) Consider a stable, first-order system whose transfer function is \( G(s) \). Let \( \beta \) denote the system’s instantaneous gain, \( \gamma \) denote the system’s steady-state gain, and \( \tau \) denote the system’s time-constant. Show that \( G(s) \) is
\[
G(s) = \frac{\beta \tau s + \gamma}{\tau s + 1}
\]
HINT: use part (b), which writes the transfer function in a specific form, and interpret the coefficients correctly.

(f) Assuming all initial conditions are 0, plot the response of \( G(s) = \frac{0.6s + 1}{0.5s + 1} \) to the input \( u \) shown below.
5. Consider the feedback system below.

There are two feedback loops: a feedback on velocity, and a feedback on position. Take \( K_P = 1 \) and \( K_D = 1.4 \). The closed-loop transfer function from \( R \to Y \) is

\[
G_{RY}^{\text{orig}} = \frac{1}{s^2 + 2(0.7)(1)s + 1}
\]

As it happens (since \( K_P = 1 \)) the closed-loop transfer function from \( D \to Y \) is also

\[
G_{DY}^{\text{orig}} = \frac{1}{s^2 + 2(0.7)(1)s + 1}
\]

Suppose we introduce a very small amount of integral control, as shown below.

If \( K_I = 0.1 \), then the transfer functions become (note: I have already computed the closed-loop transfer functions, and then for the purposes here, factored them into the product of two simpler transfer functions)

\[
G_{RY} = \frac{(1.18)(8.5)s + 1}{(8.5)s + 1} \cdot \frac{0.92^2}{s^2 + 2(0.68)(0.92)s + 0.92^2}
\]

and

\[
G_{DY} = \frac{(1.18)(8.5)s + 0}{(8.5)s + 1} \cdot \frac{0.92^2}{s^2 + 2(0.68)(0.92)s + 0.92^2}
\]

If \( K_I = 0.05 \), then the transfer functions become

\[
G_{RY} = \frac{(1.07)(18.6)s + 1}{(18.6)s + 1} \cdot \frac{0.96^2}{s^2 + 2(0.69)(0.96)s + 0.96^2}
\]

and

\[
G_{DY} = \frac{(1.07)(18.6)s + 0}{(18.6)s + 1} \cdot \frac{0.96^2}{s^2 + 2(0.69)(0.96)s + 0.96^2}
\]

If \( K_I = 0.02 \), then the transfer functions become

\[
G_{RY} = \frac{(1.03)(48.5)s + 1}{(48.5)s + 1} \cdot \frac{0.99^2}{s^2 + 2(0.699)(0.99)s + 0.99^2}
\]

and

\[
G_{DY} = \frac{(1.03)(48.5)s + 0}{(48.5)s + 1} \cdot \frac{0.99^2}{s^2 + 2(0.699)(0.99)s + 0.99^2}
\]
(a) Consider the modified $G_{RY}$. Give an interpretation of the two “simpler” systems that I have factored it into. One should be similar to the original closed-loop $G_{RY}$, and one should have a simple interpretation of its instantaneous-gain, steady-state gain, and time-constant.

(b) Consider the modified $G_{DY}$. Give an interpretation of the two “simpler” systems that I have factored it into. One should be similar to the original closed-loop $G_{DY}$, and one should have a simple interpretation of its instantaneous-gain, steady-state gain, and time-constant. What is important to notice about the steady-state gain?
(c) Shown below are the response of $y$ to unit step inputs in $r$ and $d$ respectively, as well as the magnitude of the frequency response from $d \rightarrow y$. There are 4 graphs in each - the nominal response when there was no integral control, and the response with the 3 different levels of integral control. Label the individual curves on graphs (use no integral, $K_I = 0.02$, $K_I = 0.05$ and $K_I = 0.1$ as labels).