1. (a) What is the general form of the solution to the differential equation

\[ \ddot{x}(t) + 5\dot{x}(t) + 6x(t) = 0 \]

(b) What is the general form of the solution to the differential equation

\[ \ddot{x}(t) + 5\dot{x}(t) + 6x(t) = 12 \]

Your expressions both should have two free constants.

2. Consider the complex number

\[ \gamma = \frac{3 - 4j}{8 + 6j} \]

(a) Determine \(|\gamma|\)

(b) Determine \(\angle\gamma\)

3. 12 different input\(u\)/output\(y\) systems are given below. The unit-step response, starting from zero initial conditions at \(t = 0^-\), are shown. Match the system with the step response.

(a) \[ \ddot{y}(t) + 8.4\dot{y}(t) + 36y(t) = -12\dot{u}(t) + 36u(t) \]

(b) \[ \ddot{y}(t) + 0.4\dot{y}(t) + y(t) = 5\dot{u}(t) + u(t) \]

(c) \[ \ddot{y}(t) + 0.4\dot{y}(t) + y(t) = 4\dot{u}(t) - u(t) \]

(d) \[ \ddot{y}(t) + 2\dot{y}(t) + 25y(t) = -8\dot{u}(t) + 25u(t) \]

(e) \[ \ddot{y}(t) + 8.4\dot{y}(t) + 36y(t) = -36u(t) \]

(f) \[ \ddot{y}(t) + 1.4\dot{y}(t) + y(t) = -5\dot{u}(t) - u(t) \]

(g) \[ \ddot{y}(t) + 0.4\dot{y}(t) + y(t) = -4\dot{u}(t) \]

(h) \[ \ddot{y}(t) + 8.4\dot{y}(t) + 36y(t) = -12\dot{u}(t) - 36u(t) \]

(i) \[ \ddot{y}(t) + 1.4\dot{y}(t) + y(t) = -4\dot{u}(t) + u(t) \]

(j) \[ \ddot{y}(t) + 2\dot{y}(t) + 25y(t) = 6\dot{u}(t) + 25u(t) \]

(k) \[ \ddot{y}(t) + 1.4\dot{y}(t) + y(t) = u(t) \]
4. A process, with input $u$, and output $y$, is governed by the equation

$$\ddot{y}(t) + \dot{y}(t) + y(t) = u(t)$$

A PI (Proportional plus Integral) controller is proposed

$$u(t) = K_P [r(t) - y(t)] + K_I \dot{z}(t)$$

$$\dot{z}(t) = r(t) - y(t)$$

Here $r$ is a reference input.

(a) Using the controller equations, express $\dot{u}(t)$ in terms of $r, \dot{r}, y$ and $\dot{y}$.

(b) By differentiating the process equation, and substituting, derive the closed-loop differential equation relating $r$ and $y$ (there should be no $u$ in the equation).

(c) Using the 3rd order test for stability, determine the conditions on $K_P$ and $K_I$ such that the closed-loop system is stable.

5. A process, with input $u$, disturbance $d$ and output $y$ is governed by

$$\dot{y}(t) = 2y(t) + 3u(t) + d(t)$$

(a) Is the process stable?

(b) Suppose $y(0) = 1$, and $u(t) = d(t) \equiv 0$ for all $t \geq 0$. What is the solution $y(t)$ for $t \geq 0$.
(c) Consider a proportional-control strategy, \( u(t) = K_1 r(t) + K_2 [r(t) - y(t)] \). Determine the closed-loop differential equation relating the variables \((y, r, d)\).

(d) For what values of \( K_1 \) and \( K_2 \) is the closed-loop system stable?

(e) As a function of \( K_2 \), what is the steady-state gain from \( d \to y \) in the closed-loop system?

(f) As a function of \( K_1 \) and \( K_2 \), what is the steady-state gain from \( r \to y \) in the closed-loop system?

(g) Choose \( K_1 \) and \( K_2 \) so that the steady-state gain from \( r \to y \) equals 1, and the steady-state gain from \( d \to y \) equals 0.1.

(h) With those gains chosen, sketch (try to be accurate) the two responses \( y(t) \) and \( u(t) \) for the following situation:

\[
y(0) = 0, \quad r(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq 1 \\ 1 & \text{for } 1 < t \end{cases}, \quad d(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq 2 \\ 1 & \text{for } 2 < t \end{cases}
\]
6. A 1st order process

\[ \dot{y}(t) = u(t) + d(t) \]

is controlled by a proportional control

\[ u(t) = K_P [r(t) - y_m(t)] \]

where \( y_m(t) = y(t) + n(t) \). The interpretation of signals is: \( u \) is control input; \( y \) is process output; \( d \) is external disturbance on process; \( r \) is a reference input, representing a desired value of \( y \); \( n \) is measurement noise.

(a) Eliminate \( u \) from the equations, and get the closed-loop differential equation relating \((r, d, n)\) to \( y \).

(b) Under what conditions on \( K_P \) is the closed-loop system stable?

(c) How is the time-constant of the closed-loop system related to \( K_P \)?

(d) Shown below are the closed-loop frequency responses from \((r, d, n) \rightarrow y\), as \( K_P \) increases from 0.1 to 10. Indicate on each graph with an arrow “cutting” across the plots, the direction of increasing \( K_P \).

(e) Shown below are the closed-loop frequency responses from \((r, d, n) \rightarrow u\), as \( K_P \) increases from 0.1 to 10. Indicate on each graph with an arrow “cutting” across the plots, the direction of increasing \( K_P \).