ME 132, Spring 2003, Final Exam

Name:

<table>
<thead>
<tr>
<th>#1</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
<th>#5</th>
<th>#6</th>
<th>#7</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>20</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>10</td>
<td>15</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>#8</th>
<th>#9</th>
<th>#10</th>
<th>#11</th>
<th>#12</th>
<th>#13</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>20</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>180</td>
</tr>
</tbody>
</table>

- Do problems 1-11. Also do problem 12 or problem 13, but not both. Mark above which (of 12 and 13) one you want me to count in your exam.

- Any unmarked summing junctions are positively signed (+).
1. The input \(u\), and output \(y\), of a single-input, single-output system are related by

\[ y^{[3]}(t) + 6y^{[2]}(t) + 2y^{[1]}(t) + 3y(t) = 2u^{[2]}(t) - 5u^{[1]}(t) - 5u(t) \]

(a) Find the transfer function from \(U\) to \(Y\)

(b) Show that this is a stable system.

(c) If \(u(t) \equiv 2\) for all \(t \geq 0\), what is the limiting value of \(y\), namely \(\lim_{t \to \infty}\)?

(d) Suppose the input is sinusoidal, \(u(t) = \sin(100t)\). In the steady state, what is the approximate amplitude of the sinusoidal output \(y\)?
2. Consider the 2-state system governed by the equation $\dot{x}(t) = Ax(t)$. Shown below are the phase-plane plots ($x_1(t)$ vs $x_2(t)$) for 4 different cases. Match the plots with the $A$ matrices, and correctly draw in arrows indicating the evolution in time. Put your answers on the enlarged graphs included in the solution packet.

$$A_1 = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$$
3. A feedback system is shown below.

![Feedback System Diagram]

Here $\gamma$ is a constant real number. The open-loop transfer function is $L(s)$, given as

$$L(s) = \frac{30(s + 10)(s + 10)}{(s + 1)(s + 1)(s + 1)}$$

(a) Determine the characteristic equation for the closed-loop system.

(b) Using the 3rd order test for polynomials, completely determine the range of $\gamma$ values that result in a stable system. Here is a hint, provided simply to aid you in checking your answer: I have already verified, for instance, that

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Closed-loop is</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>unstable</td>
</tr>
<tr>
<td>$-0.1$</td>
<td>unstable</td>
</tr>
<tr>
<td>$-0.0002$</td>
<td>stable</td>
</tr>
<tr>
<td>$0.001$</td>
<td>stable</td>
</tr>
<tr>
<td>$0.01$</td>
<td>unstable</td>
</tr>
<tr>
<td>$0.1$</td>
<td>stable</td>
</tr>
<tr>
<td>$1$</td>
<td>stable</td>
</tr>
</tbody>
</table>
4. In problem 3, you determined the overall possible ranges of $\gamma$ for which the system shown below is stable. As you determined (and divulged in the hint), the stability region includes the point $\gamma = 1$.

(a) Taking the nominal value of $\gamma = 1$, use the Bode plot of $L$ (provided on the next page) to determine the gain margin of the system. Show your work. Do not simply take the numbers from problem 3, and express them as a gain margin. Work out the answer independently, using the graph of $L$.

(b) Compare your answer in Problem 3 to the answer here. They are very closely related, but the gain-margin answer has “less” information. Explain.
5. Take $L$ from problem 3. There, you determined that for no time delay ($T = 0$), the closed-loop system shown below is stable.

![Block diagram of the system](image)

Use the Bode plot of $L$ (provided on the previous page) to determine the time-delay margin of the system.
6. A block diagram is shown below. $\beta$ is a real number.

![Block Diagram](image)

(a) What is the differential equation relating $y$ and $r$?

(b) What is the transfer function from $R$ to $Y$?

(c) Under what conditions (on $\beta$) is the system stable?

(d) If the system is stable, what is the time constant?

(e) If the system is stable, what is the steady-state gain from $r$ to $y$. 
7. Shown below are two systems. The system on the left is the **nominal system**, while the system on the right represents a deviation from the nominal (the insertion of the dashed box) and is called the **perturbed system**.

Based on the values of $K_P$ an $K_I$, and some analysis, you should have a general idea of how the nominal system behaves (e.g., the effect of $r$ on $u$ and $y$). Consider 3 different possibilities (listed below) regarding the relationship between the nominal and perturbed systems:

(a) The perturbed system behaves **pretty much the same** as the nominal system.
(b) The perturbed system behaves **quite differently** from the nominal system, but is still **stable**.
(c) The perturbed system is **unstable**.

For each row in the table below, which description from above applies? Write a, b, or c in each box. Show work below.

<table>
<thead>
<tr>
<th>$K_P$</th>
<th>$K_I$</th>
<th>$\beta$</th>
<th>Your Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.8</td>
<td>4</td>
<td>0.02</td>
<td></td>
</tr>
<tr>
<td>1.4</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>100</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>2500</td>
<td>0.02</td>
<td></td>
</tr>
</tbody>
</table>
8. A process, with input \( v \) and output \( y \) is governed by
\[
\dot{y}(t) - 2y(t) = v(t)
\]

(a) What is the transfer function from \( V \) to \( Y \)?

(b) Suppose \( y(0) = 1 \), and \( v(t) \equiv 0 \) for all \( t \geq 0 \). What is the solution \( y(t) \) for \( t \geq 0 \). Is the process stable?

(c) Suppose that the input \( v \) is the sum of a control input \( u \) and a disturbance input \( d \), so \( v(t) = u(t) + d(t) \). Consider a PI control strategy, \( u(t) = K_P [r(t) - y(t)] + K_I z(t), \dot{z}(t) = r(t) - y(t) \). Draw a block diagram of the closed-loop system using transfer function representations for the process and the controller. Include the external inputs \( r \) and \( d \), and label the signals \( u \) and \( y \).
(d) In the closed-loop, what are the transfer functions from $R$ to $Y$ and from $D$ to $Y$.

(e) In the closed-loop, what are the transfer functions from $R$ to $U$ and from $D$ to $U$. 
(f) For what values of $K_P$ and $K_I$ is the closed-loop system stable?

(g) Choose $K_P$ and $K_I$ so that the closed-loop characteristic equation has roots with $\xi = 0.707 (= \frac{1}{\sqrt{2}})$ and $\omega_n = 2$. 
9. The pitching-axis of a tail-fin controlled missile is governed by the nonlinear state equations

\[ \begin{align*}
\dot{\alpha}(t) &= f(\alpha(t)) \cos \alpha(t) + q(t) \\
\dot{q}(t) &= h(\alpha(t)) + Eu(t)
\end{align*} \]

Here, the states are \( x_1 := \alpha \), the angle-of-attack, and \( x_2 := q \), the angular velocity of the pitching axis. The input variable, \( u \), is the deflection of the fin which is mounted at the tail of the missile. \( E \) is a physical constant, and \( E > 0 \). \( f \) and \( h \) are known, differentiable functions (from wind-tunnel data) of \( \alpha \).

(a) Show that for any specific value of \( \bar{\alpha} \), with \( |\bar{\alpha}| < \frac{\pi}{2} \), there is a pair \((\bar{q}, \bar{u})\) such that

\[ \begin{bmatrix}
\bar{\alpha} \\
\bar{q}
\end{bmatrix}, \bar{u} \]

is an equilibrium point of the system (this represents a turn at a constant rate). Your answer should clearly show how \( \bar{q} \) and \( \bar{u} \) are functions of \( \bar{\alpha} \), and will most likely involve the functions \( f \) and \( h \).

(b) Calculate the Jacobian Linearization of the missile system about the equilibrium point. In other words, find a \( 2 \times 2 \) matrix \( A \), and a \( 2 \times 1 \) matrix \( B \), such that while they remain small, the deviation variables \( \delta_x(t) := x(t) - \bar{x}, \delta_u(t) := u(t) - \bar{u} \) are approximately governed by

\[ \dot{\delta}_x(t) = A\delta_x(t) + B\delta_u(t) \]

Your answers for \( A \) and \( B \) will be fairly symbolic, and may depend on the derivatives of the functions \( f \) and \( h \). Be sure to indicate where the various terms are evaluated.
10. The block diagram below is often called an “approximate differentiator.” Note that nowhere in the block diagram is there a differentiating element.

(a) Based on the block diagram, write the differential equation relating \( x, \dot{x} \) and \( u \).

(b) Write the equation expressing \( y \) in terms of \( x \) and \( u \).

(c) Show that the transfer function from \( U \) to \( Y \) is \( \frac{s}{\tau s + 1} \).
(d) Suppose that the initial condition is $x(0) = 0$. Apply a step input at $t = 0$, so $u(t) = \bar{u}$ for $t > 0$ (here, $\bar{u}$ is just some constant value). Compute the response $x(t)$, for $t \geq 0$.

(e) With $x(t)$ computed above, compute the output $y(t)$, and sketch below.
(f) Suppose that the initial condition is $x(0) = 0$, let $\tau = 0.2$. Apply a ramp input (with slope 3)

$$u(t) = 3t \text{ for } t \geq 0.$$ 

Compute the response $y(t)$, and plot. If you cannot derive the expression for $y$, guess what it should look like, and plot it below.
Block diagrams for two systems are shown below. Two of the blocks are just gains, \( K_P \) and \( K_D \) and the other blocks are described by their transfer functions. The constant \( \beta \) is positive, \( \beta > 0 \). The system on the left is stable if and only if \( K_P > 0 \) and \( K_D > \beta \) (no need to check this – it is correct). What are the conditions on \( K_P, K_D \) and \( \tau \), such that the system on the right is stable? **Hint:** Note that \( \tau \) is the time-constant of the filter in the approximate differentiation used to obtain \( \dot{y}_{\text{app}} \) from \( y \). The stability requirements will impose some relationship between it’s cutoff frequency \( \frac{1}{\tau} \) and the severity (eg., speed) of the unstable dynamics of the process, namely \( \beta \).
12. A block diagram is shown below. Each system is represented by its transfer function.

(a) In terms of the transfer functions of the individual blocks, what is the transfer function from $R$ to $Y$?
(b) In terms of the transfer functions of the individual blocks, what is the transfer function from $D$ to $Y$?
13. A block diagram is shown below. Each system is represented by its transfer function.

(a) In terms of $K_P, K_I, K_D, \tau$, what is the transfer function from $R$ to $Y$ (hint: the denominator should be 4th order).
(b) In terms of $K_P, K_I, K_D, \tau$, what is the transfer function from $D$ to $Y$ (hint: same as above).

(c) In terms of $K_P, K_I, K_D, \tau$, what is the characteristic polynomial of the closed-loop system?

(d) In terms of $K_P, K_I, K_D, \tau$, what is the differential equation relating $r$ and $d$ to $y$?