**Lyapunov Operator**

Let $A \in \mathbb{F}^{n \times n}$ be given, and define a linear operator $\mathcal{L}_A : \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}$ as

$$\mathcal{L}_A(X) := A^*X + AX$$

Suppose $A$ is diagonalizable (what follows can be generalized even if this is not possible - the qualitative results still hold). Let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be the eigenvalues of $A$. In general, these are complex numbers. Let $V \in \mathbb{C}^{n \times n}$ be the matrix of eigenvectors for $A^*$, so

$$V = [v_1 \ v_2 \ \cdots \ v_n] \in \mathbb{C}^{n \times n}$$

is invertible, and for each $i$

$$A^*v_i = \overline{\lambda}_iv_i$$

Let $w_i \in \mathbb{C}^n$ be defined so that

$$V^{-1} = \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_n^T \end{bmatrix}$$

Since $V^{-1}V = I_n$, we have that

$$w_i^Tv_j = \delta_{ij} := \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

For each $1 \leq i, j \leq n$ define

$$X_{ij} := v_i v_j^* \in \mathbb{C}^{n \times n}$$
Lemma 27 The set $\{X_{ij}\}_{i=1,..,n; j=1,..,n}$ is a linearly independent set, and this set is a full set of eigenvectors for the linear operator $\mathcal{L}_A$. Moreover, the eigenvalues of $\mathcal{L}_A$ is the set of complex numbers $\{\bar{\lambda}_i + \lambda_j\}_{i=1,..,n; j=1,..,n}$

Proof: Suppose $\{\alpha_{ij}\}_{i=1,..,n; j=1,..,n}$ are scalars and

$$\sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_{ij} X_{ij} = 0_{n \times n} \quad (0.8)$$

Premultiply this by $w_i^T$ and postmultiply by $\bar{w}_k$

$$0 = w_i^T \left( \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_{ij} X_{ij} \right) \bar{w}_k = \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_{ij} w_i^T v_i v_j^* \bar{w}_k = \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_{ij} \delta_{ii} \delta_{kj} = \alpha_{ik}$$

which holds for any $1 \leq k, l \leq n$. Hence, all of the $\alpha$’s are 0, and the set $\{X_{ij}\}_{i=1,..,n; j=1,..,n}$ is indeed a linearly independent set. Also, by definition of $\mathcal{L}_A$, we have

$$\mathcal{L}_A (X_{ij}) = A^* X_{ij} + X_{ij} A = A^* v_i v_j^* + v_i v_j^* A = \bar{\lambda}_i v_i v_j^* + v_i \left( \lambda_j v_j^* \right) = (\bar{\lambda}_i + \lambda_j) v_i v_j^*$$

So, $X_{ij}$ is indeed an eigenvector of $\mathcal{L}_A$ with eigenvalue $(\bar{\lambda}_i + \lambda_j)$. 

$\#$
Do a Schur decomposition of $A$, namely $A = Q\Lambda Q^*$, with $Q$ unitary and $\Lambda$ upper triangular. Then the equation $A^*X +XA = R$ can be written as

$$
A^*X +XA = R \leftrightarrow Q\Lambda^*Q^*X +XQ\Lambda Q^* = R \\
\leftrightarrow \Lambda^*Q^*XQ +Q^*XQ\Lambda = Q^*RQ \\
\leftrightarrow \Lambda^*\tilde{X} + \tilde{X}\Lambda = \tilde{R}
$$

Since $Q$ is invertible, the redefinition of variables is an invertible transformation.

Writing out all of the equations (we are trying to get that for every $\tilde{R}$, there is a unique $\tilde{X}$ satisfying the equation) yields a upper triangular system with $n^2$ unknowns, and the values $\lambda_i + \bar{\lambda}_j$ on the diagonal. Do it. ♣

This is an easy (and computationally motivated) proof that the Lyapunov operator $L_A$ is an invertible operator if and only if $\lambda_i + \bar{\lambda}_j \neq 0$ for all eigenvalues of $A$. 


If $A$ is stable (all eigenvalues have negative real parts) then $(\bar{\lambda}_i + \lambda_j) \neq 0$ for all combinations. Hence $\mathcal{L}_A$ is an invertible linear operator. We can explicitly solve the equation

$$\mathcal{L}_A (X) = -Q$$

for $X$ using an integral formula.

**Theorem 28** Let $A \in \mathbb{F}^{n \times n}$ be given, and suppose that $A$ is stable. The $\mathcal{L}_A$ is invertible, and for any $Q \in \mathbb{F}^{n \times n}$, the unique $X \in \mathbb{F}^{n \times n}$ solving $\mathcal{L}_A (X) = -Q$ is given by

$$X = \int_0^\infty e^{A^\tau} Qe^{A\tau} d\tau$$

**Proof:** For each $t \geq 0$, define

$$S(t) := \int_0^t e^{A^\tau} Qe^{A\tau} d\tau$$

Since $A$ is stable, the integrand is made up of decaying exponentials, and $S(t)$ has a well-defined limit as $t \to \infty$, which we have already denoted $X := \lim_{t \to \infty} S(t)$. For any $t \geq 0$,

$$A^* S(t) + S(t) A = \int_0^t (A^* e^{A^\tau} Qe^{A\tau} + e^{A^\tau} Qe^{A\tau} A) d\tau$$

$$= \int_0^t \frac{d}{d\tau} (e^{A^\tau} Qe^{A\tau}) d\tau$$

$$= e^{A^t} Qe^{At} - Q$$

Taking limits on both sides gives

$$A^* X + XA = -Q$$

as desired. ✷
A few other useful facts

**Theorem 29** Suppose $A, Q \in \mathbb{F}^{n \times n}$ are given. Assume that $A$ is stable and $Q = Q^* \geq 0$. Then the pair $(A, Q)$ is observable if and only if

$$X := \int_0^\infty e^{A^* \tau} Q e^{A \tau} d\tau > 0$$

**Proof:** $\Leftarrow$ Suppose not. Then there is an $x_o \in \mathbb{F}^n$, $x_o \neq 0$ such that

$$Q e^{At} x_o = 0$$

for all $t \geq 0$. Consequently $x_o^* e^{At} Q e^{At} x_o = 0$ for all $t \geq 0$. Integrating gives

$$0 = \int_0^\infty (x_o^* e^{At} Q e^{At} x_o) \, dt$$

$$= x_o^* (\int_0^\infty e^{At} Q e^{At} dt) x_o$$

$$= x_o^* X x_o$$

Since $x_o \neq 0$, $X$ is not positive definite.

$\Rightarrow$ Suppose that $X$ is not positive definite (note by virtue of the definition $X$ is at least positive semidefinite). Then there exists $x_o \in \mathbb{F}^n$, $x_o \neq 0$ such that $x_o^* X x_o = 0$. Using the integral form for $X$ (since $A$ is stable by assumption) and the fact that $Q \geq 0$ gives

$$\int_0^\infty \| Q^{1/2} e^{A \tau} x_o \| d\tau = 0$$

The integrand is nonnegative and continuous, hence it must be 0 for all $\tau \geq 0$. Hence $Q e^{A\tau} x_o = 0$ for all $\tau \geq 0$. Since $x_o \neq 0$, this implies that $(A, Q)$ is not observable. $\blacksquare$
**Theorem 30** Let \((A, C)\) be detectable and suppose \(X\) is any solution to \(A^*X + XA = -C^*C\) (there is no apriori assumption that \(L_A\) is an invertible operator, hence there could be multiple solutions). Then \(X \geq 0\) if and only if \(A\) is stable.

**Proof** $\Rightarrow$ Suppose not, then there is a \(v \in \mathbb{C}^n, v \neq 0, \lambda \in \mathbb{C}, \Re(\lambda) \geq 0\) with \(Av = \lambda v\) (and \(v^*A^* = \overline{\lambda}v^*\)). Since \((A, C)\) is detectable and the eigenvalue in the closed right-half plane \(Cv \neq 0\). Note that

\[
-\|Cv\|^2 = -v^*C^*Cv \\
= v^* (A^*X + XA) v \\
= (\overline{\lambda} + \lambda) v^*Xv \\
= 2\Re(\lambda) v^*Xv
\]

Since \(\|Cv\| > 0\) and \(\Re(\lambda) \geq 0\), we must actually have \(\Re(\lambda) > 0\) and \(v^*Xv < 0\). Hence \(X\) is not positive semidefinite.

$\Leftarrow$ Since \(A\) is stable, there is only one solution to the equation \(A^*X + XA = -C^*C\), moreover it is given by the integral formula,

\[
X = \int_0^\infty e^{A^*\tau} C^*Ce^{A\tau} d\tau
\]

which is clearly positive semidefinite. \(\dagger\)
Finally, some simple results about the ordering of Lyapunov solutions.

Let $A$ be stable, so $\mathcal{L}_A$ is guaranteed to be invertible. Use $\mathcal{L}_A^{-1}(-Q)$ to denote the unique solution $X$ to the equation $A^*X +XA = -Q$.

**Lemma 31** If $Q_1^* = Q_1 \geq Q_2 = Q_2^*$, then

$$\mathcal{L}_A^{-1}(-Q_1) \geq \mathcal{L}_A^{-1}(-Q_2)$$

**Lemma 32** If $Q_1^* = Q_1 > Q_2 = Q_2^*$, then

$$\mathcal{L}_A^{-1}(-Q_1) > \mathcal{L}_A^{-1}(-Q_2)$$

**Proofs** Let $X_1$ and $X_2$ be the two solutions, obviously

$$A^*X_1 + X_1A = -Q_1$$
$$A^*X_2 + X_2A = -Q_2$$

$$A^* (X_1 - X_2) + (X_1 - X_2) A = -(Q_1 - Q_2)$$

Since $A$ is stable, the integral formula applies, and

$$X_1 - X_2 = \int_0^\infty e^{A^*\tau} (Q_1 - Q_2) e^{A\tau} d\tau$$

Obviously, if $Q_1 - Q_2 \geq 0$, then $X_1 - X_2 \geq 0$, which proves Lemma 5. If $Q_1 - Q_2 > 0$, then the pair $(A, Q_1 - Q_2)$ is observable, and by Theorem 3, $X_1 - X_2 > 0$, proving Lemma 6. 

**Warning:** The converses of Lemma 5 and Lemma 6 are NOT true.
The algebraic Riccati equation (A.R.E.) is

\[ A^T X + X A + X R X - Q = 0 \]  

(0.9)

where \( A, R, Q \in \mathbb{R}^{n \times n} \) are given matrices, with \( R = R^T \), and \( Q = Q^T \), and solutions \( X \in \mathbb{C}^{n \times n} \) are sought. We will primarily be interested in solutions \( X \) of (0.9), which satisfy

\[ A + RX \ \text{is Hurwitz} \]

and are called *stabilizing solutions* of the Riccati equation.

Associated with the data \( A, R, \) and \( Q \), we define the *Hamiltonian* matrix \( H \in \mathbb{R}^{2n \times 2n} \) by

\[
H := \begin{bmatrix}
A & R \\
Q & -A^T
\end{bmatrix}
\]  

(0.10)
**Invariant Subspace Definition**

Suppose $A \in \mathbb{C}^{n \times n}$. Then (via matrix-vector multiplication), we view $A$ as a linear map from $\mathbb{C}^n \rightarrow \mathbb{C}^n$.

A subspace $\mathcal{V} \subset \mathbb{C}^n$ is said to be “$A$-invariant” if for every vector $v \in \mathcal{V}$, the vector $Av \in \mathcal{V}$ too.

In terms of matrices, suppose that the columns of $W \in \mathbb{C}^{n \times m}$ are a basis for $\mathcal{V}$. In other words, they are linearly independent, and span the space $\mathcal{V}$, so

$$\mathcal{V} = \{W \eta : \eta \in \mathbb{C}^m\}$$

Then $\mathcal{V}$ being $A$-invariant is equivalent to the existence of a matrix $S \in \mathbb{C}^{m \times m}$ such that

$$AW = WS$$

The matrix $S$ is the matrix representation of $A$ restricted to $\mathcal{V}$, referred to the basis $W$. Clearly, since $W$ has linearly independent columns, the matrix $S$ is unique.

If $(Q, \Lambda)$ are the Schur decomposition of $A$, then for any $k$ with $1 \leq k \leq n$, the first $k$ columns of $Q$ span a $k$-dimensional invariant subspace of $A$, as seen below

$$A [Q_1, Q_2] = [Q_1, Q_2] \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix}$$

Clearly, $AQ_1 = Q_1\Lambda_{11}$. 

Eigenvectors in Invariant subspaces

Note that since $S \in \mathbb{C}^{m \times m}$ has at least one eigenvector, say $\xi \neq 0_m$, with associated eigenvalue $\lambda$, it follows that $W\xi \in \mathcal{V}$ is an eigenvector of $A$,

$$A(W\xi) = AW\xi = WS\xi = \lambda W\xi = \lambda(W\xi)$$

Hence, every nontrivial invariant subspace of $A$ contains an eigenvector of $A$. 
If $S$ is Hurwitz, then $\mathcal{V}$ is a “stable, $A$-invariant” subspace.

Note that while the matrix $S$ depends on the particular choice of the basis for $\mathcal{V}$ (i.e., the specific columns of $W$), the eigenvalues of $S$ are only dependent on the subspace $\mathcal{V}$, not the basis choice.

To see this, suppose that the columns of $\tilde{W} \in \mathbb{C}^{n \times m}$ span the same space as the columns of $W$.

Obviously, there is a matrix $T \in \mathbb{C}^{m \times m}$ such that $W = \tilde{W} T$. The linear independence of the columns of $W$ (and $\tilde{W}$) imply that $T$ is invertible. Also,

$$A\tilde{W} = AWT^{-1} = WST^{-1} = \tilde{W} \underbrace{TST^{-1}}_{\tilde{S}}$$

Clearly, the eigenvalues of $S$ and $\tilde{S}$ are the same.

The linear operator $A|_{\mathcal{V}}$ is the transformation from $\mathcal{V} \to \mathcal{V}$ defined as $A$, simply restricted to $\mathcal{V}$.

Since $\mathcal{V}$ is $A$-invariant, this is well-defined. Clearly, $S$ is the matrix representation of this operator, using the columns of $W$ as the basis choice.
\[ A^T X + XA + XRX - Q = 0 \]

The first theorem gives a way of constructing solutions to (0.9) in terms of invariant subspaces of \( H \).

**Theorem 33** Let \( \mathcal{V} \subset \mathbb{C}^{2n} \) be a \( n \)-dimensional invariant subspace of \( H \) (\( H \) is a linear map from \( \mathbb{C}^{2n} \) to \( \mathbb{C}^{2n} \)), and let \( X_1 \in \mathbb{C}^{n \times n}, X_2 \in \mathbb{C}^{n \times n} \) be two complex matrices such that

\[ \mathcal{V} = \text{span} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \]

If \( X_1 \) is invertible, then \( X := X_2X_1^{-1} \) is a solution to the algebraic Riccati equation, and \( \text{spec} (A + RX) = \text{spec} (H | \mathcal{V}) \).

**Proof:** Since \( \mathcal{V} \) is \( H \) invariant, there is a matrix \( \Lambda \in \mathbb{C}^{n \times n} \) with

\[ \begin{bmatrix} A & R \\ Q & -A^T \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \Lambda \]  

(0.11)

which gives

\[ AX_1 + RX_2 = X_1 \Lambda \]  

(0.12)

and

\[ QX_1 - A^T X_2 = X_2 \Lambda \]  

(0.13)

Pre and post multiplying equation (0.12) by \( X_2X_1^{-1} \) and \( X_1^{-1} \) respectively gives

\[ (X_2X_1^{-1}) A + (X_2X_1^{-1}) R (X_2X_1^{-1}) = X_2 \Lambda X_1^{-1} \]  

(0.14)

and postmultiplying 0.13 by \( X_1^{-1} \) gives

\[ Q - A^T (X_2X_1^{-1}) = X_2 \Lambda X_1^{-1} \]  

(0.15)

Combining (0.14) and (0.15) yields

\[ A^T (X_2X_1^{-1}) + (X_2X_1^{-1}) A + (X_2X_1^{-1}) R (X_2X_1^{-1}) - Q = 0 \]

so \( X_2X_1^{-1} \) is indeed a solution. Equation (0.12) implies that \( A + R (X_2X_1^{-1}) = X_1 \Lambda X_1^{-1} \), and hence they have the same eigenvalues. But, by definition, \( \Lambda \) is a matrix representation of the map \( H | \mathcal{V} \), so the theorem is proved. ♦
$A^TX + XA + XRX - Q = 0$

Invariant Subspace

The next theorem shows that the specific choice of matrices $X_1$ and $X_2$ which span $\mathcal{V}$ is not important.

**Theorem 34** Let $\mathcal{V}, X_1$, and $X_2$ be as above, and suppose there are two other matrices $\tilde{X}_1, \tilde{X}_2 \in \mathbb{C}^{n \times n}$ such that the columns of $\begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix}$ also span $\mathcal{V}$. Then $X_1$ is invertible if and only if $\tilde{X}_1$ is invertible. Consequently, both $X_2X_1^{-1}$ and $\tilde{X}_2\tilde{X}_1^{-1}$ are solutions to (0.9), and in fact they are equal.

**Remark:** Hence, the question of whether or not $X_1$ is invertible is a property of the subspace, and not of the particular basis choice for the subspace. Therefore, we say that an $n$-dimensional subspace $\mathcal{V}$ has the invertibility property if in any basis, the top portion is invertible. In the literature, this is often called the complimentary property.

**Proof:** Since both sets of columns span the same $n$-dimensional subspace, there is an invertible $K \in \mathbb{C}^{n \times n}$ such that

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix} K$$

Obviously, $X_1$ is invertible if and only if $\tilde{X}_1$ is invertible, and $X_2X_1^{-1} = (\tilde{X}_2K)(\tilde{X}_1K)^{-1} = \tilde{X}_2\tilde{X}_1^{-1}$. ♦.
\[ A^T X + X A + X R X - Q = 0 \]

As we would hope, the converse of theorem 33 is true.

**Theorem 35** If \( X \in \mathbb{C}^{n \times n} \) is a solution to the A.R.E., then there exist matrices \( X_1, X_2 \in \mathbb{C}^{n \times n} \), with \( X_1 \) invertible, such that \( X = X_2 X_1^{-1} \) and the columns of \( \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \) span an \( n \)-dimensional invariant subspace of \( H \).

**Proof:** Define \( \Lambda := A + RX \in \mathbb{C}^{n \times n} \). Multiplying this by \( X \) gives \( X \Lambda = X A + X R X = Q - A^T X \), the second equality coming from the fact that \( X \) is a solution to the Riccati equation. We write these two relations as

\[
\begin{bmatrix} A & R \\ Q & -A^T \end{bmatrix} \begin{bmatrix} I_n \\ X \end{bmatrix} = \begin{bmatrix} I_n \\ X \end{bmatrix} \Lambda
\]

Hence, the columns of \( \begin{bmatrix} I_n \\ X \end{bmatrix} \) span an \( n \)-dimensional invariant subspace of \( H \), and defining \( X_1 := I_n \), and \( X_2 := X \) completes the proof. \&.

Hence, there is a one-to-one correspondence between \( n \)-dimensional invariants subspaces of \( H \) with the invertibility property, and solutions of the Riccati equation.
\[ A^T X + X A + X R X - Q = 0 \]

Eigenvalue distribution of \( H \)

In the previous section, we showed that any solution \( X \) of the Riccati equation is intimately related to an invariant subspace, \( \mathcal{V} \), of \( H \), and the matrix \( A + RX \) is similar (ie. related by a similarity transformation) to the restriction of \( H \) on \( \mathcal{V} \), \( H|_\mathcal{V} \). Hence, the eigenvalues of \( A + RX \) will always be a subset of the eigenvalues of \( H \). Recall the particular structure of the Hamiltonian matrix \( H \),

\[
H := \begin{bmatrix}
A & R \\
Q & -A^T
\end{bmatrix}
\]

Since \( H \) is real, we know that the eigenvalues of \( H \) are symmetric about the real axis, including multiplicities. The added structure of \( H \) gives additional symmetry in the eigenvalues as follows.

**Lemma 36** The eigenvalues of \( H \) are symmetric about the origin (and hence also about the imaginary axis).

**Proof:** Define \( J \in \mathbb{R}^{2n \times 2n} \) by

\[
J = \begin{bmatrix}
0 & -I \\
I & 0
\end{bmatrix}
\]

Note that \( J H J^{-1} = -H^T \). Also, \( J H = -H^T J = (JH)^T \). Let \( p(\lambda) \) be the characteristic polynomial of \( H \). Then

\[
p(\lambda) := \det(\lambda I - H) = \det(\lambda I - JHJ^{-1}) = \det(\lambda I + H^T) = \det(\lambda I + H) = (-1)^{2n} \det(-\lambda I - H) = p(-\lambda)
\]

Hence the roots of \( p \) are symmetric about the origin, and the symmetry about the imaginary axis follows. ♦

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Lemma 37 If $H$ has no eigenvalues on the imaginary axis, then $H$ has $n$ eigenvalues in the open-right-half plane, and $n$ eigenvalues in the open-left-half plane.

Theorem 38 Suppose $\mathcal{V}$ is a $n$-dimensional invariant subspace of $H$, and $X_1, X_2 \in \mathbb{C}^{n \times n}$ form a matrix whose columns span $\mathcal{V}$. Let $\Lambda \in \mathbb{C}^{n \times n}$ be a matrix representation of $H|_{\mathcal{V}}$. If $\lambda_i + \lambda_j \neq 0$ for all eigenvalues $\lambda_i, \lambda_j \in \text{spec}(\Lambda)$, then $X_1^* X_2 = X_2^* X_1$.

Proof: Using the matrix $J$ from the proof of Lemma 36 we have that
\[
\begin{bmatrix} X_1^* & X_2^* \end{bmatrix} J H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}
\]
is Hermitian, and equals $(-X_1^* X_2 + X_2^* X_1) \Lambda$. Define $W := -X_1^* X_2 + X_2^* X_1$. Then $W = -W^*$, and $W \Lambda + \Lambda^* W = 0$. By the eigenvalue assumption on $\Lambda$, the linear map $\mathcal{L}_\Lambda: \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}$ defined by
\[
\mathcal{L}_\Lambda (X) = X \Lambda + \Lambda^* X
\]
is invertible, so $W = 0$. ♦

Corollary 39 Under the same assumptions as in theorem 38, if $X_1$ is invertible, then $X_2 X_1^{-1}$ is Hermitian.
\[ A^T X + X A + X R X - Q = 0 \]  

Real Solutions

Real (rather than complex) solutions arise when a symmetry condition is imposed on the invariant subspace.

**Theorem 40** Let \( \mathcal{V} \) be a \( n \)-dimensional invariant subspace of \( H \) with the invertibility property, and let \( X_1, X_2 \in \mathbb{C}^{n \times n} \) form a \( 2n \times n \) matrix whose columns span \( \mathcal{V} \). Then \( \mathcal{V} \) is conjugate symmetric (\( \{ \bar{v} : v \in \mathcal{V} \} = \mathcal{V} \)) if and only if \( X_2 X_1^{-1} \in \mathbb{R}^{n \times n} \).

**Proof:** ← Define \( X := X_2 X_1^{-1} \). By assumption, \( X \in \mathbb{R}^{n \times n} \), and

\[
\text{span} \begin{bmatrix} I_n \\ X \end{bmatrix} = \mathcal{V}
\]

so \( \mathcal{V} \) is conjugate symmetric.

→ Since \( \mathcal{V} \) is conjugate symmetric, there is an invertible matrix \( K \in \mathbb{C}^{n \times n} \) such that

\[
\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} K
\]

Therefore, \( \overline{(X_2 X_1^{-1})} = \overline{X_2 X_1^{-1}} = X_2 K (X_1 K)^{-1} = X_2 X_1^{-1} \) as desired. ♦

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\[ A^T X + X A + X R X - Q = 0 \]

Stabilizing Solutions

Recall that for any square matrix \( M \), there is a “largest” stable invariant subspace. That is, there is a stable invariant subspace \( V^*_S \) that contains every other stable invariant subspace of \( M \). This largest stable invariant subspace is simply the span of all of the eigenvectors and generalized eigenvectors associated with the open-left-half plane eigenvalues. If the Schur form has the eigenvalues (on diagonal of \( \Lambda \)) ordered, then it is spanned by the “first” set of columns of \( Q \).

**Theorem 41** There is at most one solution \( X \) to the Riccati equation such that \( A + RX \) is Hurwitz, and if it does exist, then \( X \) is real, and symmetric.

**Proof:** If \( X \) is such a stabilizing solution, then by theorem 35 it can be constructed from some \( n \)-dimensional invariant subspace of \( H \), \( V \). If \( A + RX \) is stable, then we must also have \( \text{spec}(H|_V) \). Recall that the spectrum of \( H \) is symmetric about the imaginary axis, so \( H \) has at most \( n \) eigenvalues in \( C_- \), and the only possible choice for \( V \) is the stable eigenspace of \( H \). By theorem 34, every solution (if there are any) constructed from this subspace will be the same, hence there is at most 1 stabilizing solution.

Associated with the stable eigenspace, any \( \Lambda \) as in equation (0.11) will be stable. Hence, if the stable eigenspace has the invertibility property, from corollary 39 we have \( X := X_2X_1^{-1} \) is Hermitian. Finally, since \( H \) is real, the stable eigenspace of \( H \) is indeed conjugate symmetric, so the associated solution \( X \) is in fact real, and symmetric.
If $H$ is (0.10) has no imaginary axis eigenvalues, then $H$ has a unique $n$-dimensional stable invariant subspace $\mathcal{V}_S \subset \mathbb{C}^{2n}$, and since $H$ is real, there exist matrices $X_1, X_2 \in \mathbb{R}^{n \times n}$ such that

$$\text{span} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \mathcal{V}_S.$$ 

More concretely, there exists a matrix $\Lambda \in \mathbb{R}^{n \times n}$ such that

$$H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \Lambda, \quad \text{spec}(\Lambda) \subset \mathbb{C}^-.$$ 

This leads to the following definition

**Definition 42 (Definition of $\text{dom}_S\text{Ric}$)** Consider $H \in \mathbb{R}^{2n \times 2n}$ as defined in (0.10). If $H$ has no imaginary axis eigenvalues, and the matrices $X_1, X_2 \in \mathbb{R}^{n \times n}$ for which

$$\text{span} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

is the stable, $n$-dimensional invariant subspace of $H$ satisfy $\det (X_1) \neq 0$, then $H \in \text{dom}_S\text{Ric}$. For $H \in \text{dom}_S\text{Ric}$, define $\text{Ric}_S (H) := X_2X_1^{-1}$, which is the unique matrix $X \in \mathbb{R}^{n \times n}$ such that $A + RX$ is stable, and

$$A^T X + X A + XRX - Q = 0.$$ 

Moreover, $X = X^T$. 

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Theorem 43 Suppose $H$ (as in (0.10)) does not have any imaginary axis eigenvalues, $R$ is either positive semidefinite, or negative semidefinite, and $(A, R)$ is a stabilizable pair. Then $H \in \text{dom}_S\text{Ric}$.

Proof: Let $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ form a basis for the stable, $n$-dimensional invariant subspace of $H$ (this exists, by virtue of the eigenvalue assumption about $H$). Then

\[
\begin{bmatrix} A & R \\ Q & -A^T \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \Lambda \tag{0.17}
\]

where $\Lambda \in \mathbb{C}^{n \times n}$ is a stable matrix. Since $\Lambda$ is stable, it must be that $X_1^*X_2 = X_2^*X_1$. We simply need to show $X_1$ is invertible. First we prove that $\ker X_1$ is $\Lambda$-invariant, that is, the implication

\[ x \in \ker X_1 \rightarrow \Lambda x \in \ker X_1 \]

Let $x \in \ker X_1$. The top row of (0.17) gives $RX_2 = X_1\Lambda - AX_1$. Premultiply this by $x^*X_2^*$ to get

\[
x^*X_2^*RX_2x = x^*X_2^*(X_1\Lambda - AX_1)x \\
= x^*X_2^*X_1\Lambda x \\
= x^*X_1^*X_2\Lambda x \quad \text{since} \quad X_2^*X_1 = X_1^*X_2 \Rightarrow \\
= 0
\]

Since $R \geq 0$ or $R \leq 0$, it follows that $RX_2x = 0$. Now, using (0.17) again gives $X_1\Lambda x = 0$, as desired. Hence $\ker X_1$ is a subspace that is $\Lambda$-invariant. Now, if $\ker X_1 \neq \{0\}$, then there is a vector $v \neq 0$ and $\lambda \in \mathbb{C}, \text{Re} (\lambda) < 0$ such that

\[
\begin{align*}
X_1v &= 0 \\
\Lambda v &= \lambda v \\
RX_2v &= 0
\end{align*}
\]
We also have $QX_1 - A^T X_2 = X_2 \Lambda$, from the second row of (0.17). Mutiplying by $v^*$ gives

$$v^* X_2^* [A - (\bar{\lambda} I) R] = 0$$

Since $\text{Re}(\bar{\lambda}) < 0$, and $(A, R)$ is assumed stabilizable, it must be that $X_2 v = 0$. But $X_1 v = 0$, and

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

is full-column rank. Hence, $v = 0$, and $\text{Ker} X_1$ is indeed the trivial set $\{0\}$. This implies that $X_1$ is invertible, and hence $H \in \text{dom}_5 \text{Ric}$. #.
The next theorem is a main result of LQR theory.

**Theorem 44** Let $A, B, C$ be given, with $(A, B)$ stabilizable, and $(A, C)$ detectable. Define

$$H := \begin{bmatrix} A & -BB^T \\ -C^T C & -A^T \end{bmatrix}$$

Then $H \in \text{dom}_S \text{Ric}$. Moreover, $0 \leq X := \text{Ric}_S(H)$, and

$$\text{Ker} X \subset \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

**Proof:** Use stabilizablity of $(A, B)$ and detectability of $(A, C)$ to show that $H$ has no imaginary axis eigenvalues. Specifically, suppose $v_1, v_2 \in \mathbb{C}^n$ and $\omega \in \mathbb{R}$ such that

$$\begin{bmatrix} A & -BB^T \\ -C^T C & -A^T \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = j\omega \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Premultiply top and bottom by $v_2^*$ and $v_1^*$, combine, and conclude $v_1 = v_2 = 0$. Show that if $(A, B)$ is stabilizable, then $(A, -BB^T)$ is also stabilizable. At this point, apply Theorem 43 to conclude $H \in \text{dom}_S \text{Ric}$. Then, rearrange the Riccati equation into a “Lyapunov” equation and use Theorem 28 to show that $X \geq 0$. Finally, show that $\text{Ker} X \subset \text{Ker} C$, and that $\text{Ker} X$ is $A$-invariant. That shows that $\text{Ker} X \subset \text{Ker} CA^j$ for any $j \geq 0$. \#.