1. In this problem, we will explore the bandwidth limitations that arise due to a RHP zero in the plant. Assume that the plant is strictly proper, \( P(\infty) = 0 \), and that there is a right-half-plane zero, \( z \). So, \( P(z) = 0 \), and for simplicity, assume that \( z \in \mathbb{R} \), and \( z > 0 \). Suppose that we are concerned with the sensitivity transfer function \( S = \frac{1}{1 + PC} \), which is the transfer function from \( d \to y \), shown below.

Suppose that we have constants \( \alpha, \beta \), with \( 0 < \beta < 1 < \alpha \). Also, suppose that a stabilizing controller has been designed so that for a unit-step for \( d \), the signal \( y \) satisfies

\[
|y(t)| \leq \alpha \text{ for } t \in [0, T] \quad \text{and} \quad |y(t)| \leq \beta \text{ for } t > T
\]

(a) Draw a picture showing the time-domain constraints listed above.

(b) Since the closed-loop is stable, can \( C \) have a pole at \( s = z \)?

(c) What must be the value of \( S(s)\big|_{s = z} \)?

(d) Let \( E(s) \) be the Laplace transform of the signal \( y \) (due to the unit step in \( d \)). What is the value of \( E(s)\big|_{s = z} \)?

(e) Using the general fact from calculus that for a function \( f \),

\[
\left| \int_a^b f(\tau)d\tau \right| \leq \int_a^b |f(\tau)| d\tau
\]

show that it must be that

\[
T \geq \frac{1}{z \ln \frac{\alpha - \beta}{\alpha - 1}}
\]

(f) Holding any two of the variables (from \( \alpha, \beta \) and \( z \)) fixed, how is this lower bound dependent on the other one? What does this imply about the effect of RHP zeros on the achievable speed of response?

(g) How would you modify this for \( z \in \mathbb{C} \)?

(h) Modify the problem in a natural way to get similar type bounds for the case \( \text{Re}(z) = 0 \)?

2. We now repeat problem (1) from a frequency domain viewpoint. In this problem, we will explore the bandwidth limitations that arise due to a RHP zero in the plant. Assume that the plant is strictly proper, \( P(\infty) = 0 \). Suppose that we are
trying to design a controller such that the sensitivity function lies below the bound shown below:

Note that we are asking for

- a reduction (over open loop) in the sensitivity function for all frequencies up to \( \omega_B \)
- a worst case peak of 2 for \( \|S\|_\infty \), and in fact, this cannot occur before \( \omega_P \).
- below \( \omega_B \), and until \( \omega_L \) we want the sensitivity reduction to improve, the bound dropping off at 20db/dec.
- at least a level of \( 0.5\omega_P/\omega_L \) reduction for all frequencies in \([0 \omega_L]\).

(a) In terms of \( \omega_P \) and \( \omega_L \), find a stable, minimum phase weighting function \( W(s) \), whose straight-line Bode magnitude approximation is equal to the reciprocal of the bound shown above. Define \( \omega_{B,a} \) to be the actual frequency where this weight has magnitude = 1. Note that this is slightly different from \( \omega_B \).

(b) Suppose that the plant has a zero, \( z \), in the closed-right-half plane. Suppose that there is a stabilizing controller \( C \) such that \( \|WS\|_\infty < 1 \).

i. Suppose that the zero is real, ie., \( z \in \mathbb{R} \). Show that we must have

\[
z > \sqrt{3\omega_{B,a}^2 + 4\omega_L^2} - 2\omega_L
\]

Also show that in the limit of \( \omega_L \to 0 \) (ie., perfect tracking at zero frequency) the zero and bandwidth must satisfy

\[
\omega_{B,a} < \frac{1}{\sqrt{3}}z
\]

ii. Suppose that the zero is imaginary, ie., \( z = j\omega_z \) for some real \( \omega_z \). Show that we must have

\[
\omega_{B,a} < |z|
\]
iii. Finally, suppose that the zero is complex. Find an inequality that must be satisfied involving both $|z|$ and $\text{Real}(z)$, as well as the bandwidth $\omega_{B,a}$.

3. We stated the maximum modulus theorem for stable, proper rational transfer functions. Given that result, it is easy to prove a maximum modulus theorem for matrix transfer functions. Suppose $G \in S_{n \times m}$. Show that

$$\max_{\omega \in \mathbb{R}} \bar{\sigma}(G(j\omega)) = \max_{\text{Re}(s) \geq 0} \bar{\sigma}(G(s))$$

Hint: Recall that for $M \in \mathbb{C}^{n \times m}$

$$\bar{\sigma}[M] = \max_{u \in \mathbb{C}^{n}, v \in \mathbb{C}^{m}} |u^* M v|$$

\[u\|_2=\|v\|_2=1\]

4. Suppose (for simplicity) that the plant $P$ is stable, and that its transfer function $P \in S_{n \times n}$. Assume that the plant has a right-half plane zero at $z \in \mathbb{C}$, so that $P(z)$ is singular matrix. Suppose $w \in S$ is a scalar, stable transfer function. Show that for any stabilizing controller $C$, it must be that

$$\max_{\omega \in \mathbb{R}} \bar{\sigma}[w(j\omega)(I + P(j\omega)C(j\omega))^{-1}] \geq |w(z)|$$

Hint: In the SISO case, we know that for any RHP zero of $P$ and for any stabilizing controller $C$, it must be that

$$\left.\frac{1}{1 + PC}\right|_{s = z} = 1$$

You need to show some analogous result for MIMO systems. I have made it simple here by assuming that $P$ is stable. Hence (from earlier HW), the possible sensitivity functions $(I + PC)^{-1}$ are given by $I - PQ$, where $Q$ is restricted (and allowed) to be any stable system.

5. Consider two plants

$$P_1(s) = \frac{-1s + 1}{s - 1}, \quad P_2(s) = \frac{-3s + 1}{s - 3}$$

Quantify in what ways one plant is harder to control than the other.