1. This problem is to recall some facts and formulae you already know.

(a) Let $A$ and $B$ be matrices of appropriate dimension. Show that $(A, B)$ is controllable if and only if

\[ W(t) := \int_0^t e^{A\tau} BB^* e^{A^*\tau} d\tau \]

is invertible for all $t > 0$.

(b) Verify that for any $T > 0$

\[ \int_0^T e^{A(T-\tau)} BB^* e^{A^*(T-\tau)} d\tau = \int_0^T e^{A\tau} BB^* e^{A^*\tau} d\tau \]

(c) Suppose $q \in \mathbb{R}^n$, and consider the input

\[ u(t) := B^* e^{A^*(T-t)} q \quad \text{for} \quad 0 \leq t \leq T \]

\[ u(t) = 0 \quad \text{for} \quad t > T \]

for the linear system $\dot{x}(t) = Ax(t) + Bu(t)$. Starting from $x(0) = 0$, show that $x(T) = W(T)q$.

(d) Assuming controllability, derive an input signal $u$ which transfers the initial condition $x(0) = 0$ to a desired final condition $x(T) = x_{\text{des}}$.

(e) Note that since the system is time-invariant, this input transfers the initial condition $x(-T) = 0$ to a desired final condition $x(0) = x_{\text{des}}$.

2. State-space calculation of Hankel norm:

Notation: $\mathcal{L}_2$ denotes square-integrable functions on the space $(-\infty, \infty)$. $\mathcal{L}_2^+$ denotes square-integrable functions on the space $[0, \infty)$. $\mathcal{L}_2^-$ denotes square-integrable functions on the space $(-\infty, 0)$.

For a signal $g \in \mathcal{L}_2$, define a projection operator $P^+$ which maps $\mathcal{L}_2$ into $\mathcal{L}_2^+$ via

\[ (P^+ g)(t) := g(t) \quad \text{for} \quad t \geq 0 \]

For a function $f$ defined on $(-\infty, 0]$, use the notation

\[ \|f\|_2^2 := \int_{-\infty}^0 f^*(t)f(t) \, dt, \]

if the integral is finite, and write $f \in \mathcal{L}_2^-$. Similar notation for functions defined on $[0, \infty)$.
Consider a **stable** linear system, with state space data $A$, $B$, and $C$.

$$\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*} \tag{1}$$

Assume that $(A, B)$ is controllable. Let $X_C$ and $X_O$ respectively denote the controllability and observability grammians. That is, $X_C$ is the unique solution to the linear equation

$$AX_C + X_C A^* = -BB^*,$$

while $X_O$ is the unique solution to the linear equation

$$A^*X_O + X_O A = -C^*C.$$

(a) Consider the free system

$$\begin{align*}
\dot{x} &= Ax \\
y &= Cx \\
x(0) &= x_o
\end{align*}$$

Show that the 2-norm of the resulting output $y \in L_2^+$ satisfies

$$\|y\|_2^2 := \int_0^\infty y^*(t)y(t) \, dt = x_o^*X_Ox_o$$

(b) Now consider the driven system,

$$\dot{x} = Ax + Bu$$

and assume that the pair $(A, B)$ is controllable. Hence, $X_C$ is positive definite. Let $x_0 \in \mathbb{R}^n$ (or $\mathbb{C}^n$ if we are considering complex systems). Consider the problem of determining the minimum 2-norm input $u \in L_2^-$, such that

$$\lim_{t \to -\infty} x(t) = 0 , \quad x(0) = x_o$$

Specifically, consider the optimization

$$J(x_o) := \min_{t \to -\infty} \int_0^\infty u^*(t)u(t) \, dt$$

subject to

$$\lim_{t \to -\infty} x(t) = 0 , \quad x(0) = x_o$$

Show that $J(x_o) = x_o^*X_C^{-1}x_o$. **Hint:** Assume that the input $u$ satisfies the constraints, and show that

$$\|u\|_2^2 = x_o^*X_C^{-1}x_o + \|u - B^*X_C^{-1}x\|_2^2$$

You can show this by computing

$$\int_{-\infty}^0 \frac{d}{d\tau} \left[ x^*(\tau)X_C^{-1}x(\tau) \right] \, d\tau$$

along trajectories of the linear system.
(c) Consider a state-coordinate change, \( \tilde{x} = T^{-1}x \), for an invertible matrix \( T \). Hence \( \tilde{A} = T^{-1}AT, \tilde{B} = T^{-1}B \) and \( \tilde{C} = CT \). Find the controllability and observability grammians \( \tilde{X}_C, \tilde{X}_O \) in these new coordinates.

(d) Suppose that \((A, B)\) is controllable, and \((A, C)\) is observable. Define a transformation \( T \in \mathbb{C}^{n \times n} \) using the following steps:

i. Let \( R \in \mathbb{C}^{n \times n} \) be the Hermitian, positive definite square root of \( X_O \), so \( R = R^* > 0 \), and \( X_O = R^2 \) (actually, \( R \) need not be Hermitian, just need to satisfy \( RR^* = X_O \), so I will write all of the further definitions as though \( R \) is not Hermitian)

ii. Do a Hermitian eigenvalue decomposition of the positive definite matrix \( R^*X_C R =: U \Sigma^2 U^* \), where \( U \) is a unitary matrix, and \( \Sigma \) is a diagonal, positive definite matrix. Assume that the diagonal elements of \( \Sigma \) have been rearranged in descending order.

iii. How are the eigenvalues of \( X_C X_O \) related to the entries of \( \Sigma \).

iv. Define \( T := R^{-*}U\Sigma^{1/2} \)

Show that

\[
T^* X_O T = T^{-1} X_C T^{-*} = \Sigma
\]

If we transform the original state-space matrices using this transformation, the controllability grammians and observability grammians become equal, and diagonal. This type of realization is called \textbf{balanced}.

(e) Suppose that we use the coordinate transform defined above, and change the state coordinates so that the observability and controllability grammians are identical, and in fact, diagonal. Assume that \( \sigma_1 >> \sigma_n \). Under that assumption, in what sense is the state

\[
\begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

more important than the state

\[
\begin{bmatrix}
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\]

(f) Find the relationship between the eigenvalues of \( X_C X_O \) and the eigenvalues of \( \tilde{X}_C \tilde{X}_O \).

(g) Show that

\[
\max_{u \in L^2_2} \left\| \begin{bmatrix} \mathbf{x}^* X_0 \mathbf{x} \\
\mathbf{x}^* \mathbf{x} \end{bmatrix} \right\|_2 \leq \max_{\mathbf{x} \in \mathbb{C}^{n \times 1}, \|\mathbf{x}\|_2 = 1} \frac{\mathbf{x}^* X_0 \mathbf{x}}{\|\mathbf{x}\|_2} \leq \max_{\mathbf{x} \in \mathbb{C}^{n \times 1}, \|\mathbf{x}\|_2 = 1} \frac{\mathbf{x}^* \left( X_C^{-1} \right) \mathbf{x}}{\|\mathbf{x}\|_2} \]
(h) In a rough sense, the quantity

\[
\max_{\|u\| = 1} \|P^+y\|_2^2
\]

is the induced norm from “past inputs” to “future outputs”. It is called
the Hankel Norm of the system described by the triple \((A, B, C)\). Note by
Homework problem 4, it can be computed by solving two linear equations,
and then computing the largest eigenvalue of a product of 2 positive definite
matrices.

3. Let \(V_1 \subset \mathbb{C}^n\) and \(V_2 \subset \mathbb{C}^n\) be invariant subspaces of a matrix \(H \in \mathbb{C}^{n \times n}\).

(a) Show that \(V_1 \cap V_2\) is an invariant subspace of \(H\).

(b) Show that \(V_1 + V_2\) is an invariant subspace of \(H\).

(c) Suppose that \(V_1\) and \(V_2\) are stable invariant subspaces of \(H\). This means that
both are invariant subspaces, and if \(V_1\) and \(V_2\) are matrices whose columns
form a basis for the respective subspaces, then \(AV_i = V_i \Lambda_i\) for some Hurwitz
matrices \(\Lambda_1\) and \(\Lambda_2\). Show that \(V_1 + V_2\) is a stable invariant subspace of \(H\).
(\textbf{Hint:} one way to do this is to choose basis for all of the subspaces, and
write out matrix representations of \(H\) in these basis).

(d) Suppose \(H\) has \(l \leq n\) eigenvalues in the open-left-half plane. Show that there
is a unique \(l\)-dimensional, stable invariant subspace of \(H\). \textbf{Hints:}

i. There is an \(l\)-dimensional, stable invariant subspace, spanned by the
eigenvectors and generalized eigenvectors associated with the stable eigenvalues – similarly, do Schur decomposition with the stable eigenvalues
before the right-half plane ones, and simply pick the first \(l\) columns of
the unitary matrix.

ii. There is not an \(l + 1\)-dimensional stable invariant subspace, because that
would imply that \(H\) has at least \(l + 1\) eigenvalues in the open-left-half
plane;

iii. If there are two different \(l\)-dimensional, stable, invariant subspaces, then
their subspace sum is also a stable invariant subspace, but of dimension
\(> l\), leading to a contradiction).

4. Suppose \(X, Y \in \mathbb{C}^{n \times n}\), with \(X = X^* > 0\), and \(Y = Y^* \geq 0\). Show that

\[
\max_{x \in \mathbb{C}^n, \|x\|_2 = 1} \frac{x^*Yx}{x^*X^{-1}x} = \max_{x \in \mathbb{C}^n, x \neq 0_n} \frac{x^*Yx}{x^*X^{-1}x} = \lambda_{\max}\left(X^{1/2}YX^{1/2}\right) = \rho(XY)
\]

What if \(\mathbb{C}\) is replaced by \(\mathbb{R}\), everywhere?
5. Consider the linear, time-invariant system
\[ \dot{x}(t) + Ax(t) + Bu(t) \]
with \( x(t) \in \mathbb{R}^n \), and \( u(t) \in \mathbb{R}^m \). Suppose that \( Q \in \mathbb{R}^{n \times n} \), with \( Q = Q^T \geq 0 \). Assume that \((A, B)\) is stabilizable, and \((A, Q^{1/2})\) is detectable. Using a “completion of squares” approach, determine the value of the function \( J : \mathbb{R}^n \rightarrow \mathbb{R} \)
\[ J(x_0) := \min_{u \in L_2} \int_0^\infty \left[ x^T(t)Qx(t) + u^T(t)u(t) \right] dt \]
subject to
\[ x(0) = x_0 \]
\[ \dot{x}(t) = Ax(t) + Bu(t) \]
Also determine the input \( u \) which achieves the optimum.

6. Suppose \( G \) has \( \|G\|_\infty < 1 \). Find a single quadratic Lyapunov function which verifies \( S(G, \Delta) \) is stable for all \( \Delta \in \mathbb{C}^{n_u \times n_y} \) with \( \sigma(\Delta) \leq 1 \).

7. Suppose \( G \) has \( \|G\|_\infty < 1 \) and \( \Delta \) has \( \|\Delta\|_\infty \leq 1 \). Find a quadratic Lyapunov function, using sums of quadratic Lyapunov functions for \( G \) and \( \Delta \) which verifies that \( S(G, \Delta) \) is stable.

8. Suppose \( A \) is square, and has all of its eigenvalues equal to \( \alpha \neq 0 \), and only one eigenvector (so \( A \) has only one Jordan block). Find the eigenvectors, generalized eigenvectors, and eigenvalues of the Lyapunov operator \( L_A \).

9. **State-space calculation of \( \|\cdot\|_2 \) norm:** Consider a stable linear system, with state space data \( A, B, \) and \( C \).
\[ \dot{x} = Ax + Bu \]
\[ y = Cx \]
Let \( X_C \) and \( X_O \) respectively denote the controllability and observability gramians. Show that
\[ \int_0^\infty Ce^{At}BB^*e^{A^*t}C^*dt = CX_C \]
and
\[ \text{trace} \left( \int_0^\infty Ce^{At}BB^*e^{A^*t}C^*dt \right) = \text{trace} \left( CX_C \right) = \text{trace} \left( B^*X_OB \right) \]
How is this related to the frequency domain quantities
\[ \text{trace} \left[ \int_{-\infty}^\infty G^*(j\omega)G(j\omega)d\omega \right], \quad \text{trace} \left[ \int_{-\infty}^\infty G(j\omega)G^*(j\omega)d\omega \right] \]
10. **State-space calculation of $\|\cdot\|_\infty$ norm:** Consider a linear system

$$\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{align*}$$

In this analysis, it is not necessary to assume that $A$ is stable, but we must assume that $A$ has no imaginary-axis eigenvalues. Introduce the notation $G^a(s) := [G(-\bar{s})]^*$. Easy manipulation (check this) shows that

$$M(s) := [I - G(s)G^a(s)]^{-1} = \begin{bmatrix}
A & BB^* \\
-C^*C & -A^* \\
C & 0 \\
0 & I
\end{bmatrix}$$

(a) Consider a given frequency $\bar{\omega}$. Show that $G(j\bar{\omega})$ has a singular value equal to 1 (some singular value - not necessarily the maximum) if and only if $I - G(s)G^a(s)$ is singular at $s = j\bar{\omega}$.

(b) Consider a given frequency $\bar{\omega}$. Show that $G(j\bar{\omega})$ has a singular value equal to 1 if and only if $M(s)$ has a pole at $s = j\bar{\omega}$.

(c) Hence, the imaginary axis poles of $M(s)$ are the same as the points on the imaginary axis where $G$ has a singular value equal to 1. In this part, we will show that the imaginary-axis poles of $M(s)$ are exactly equal to the imaginary axis eigenvalues of the “$A$” matrix for $M$ (it could be possible that some eigenvalues of the “$A$” matrix are uncontrollable and/or unobservable, so they would not show up in the transfer function - this calculation will rule that possibility out). To do this, show that any imaginary-axis eigenvalues of

$$\begin{bmatrix}
A & BB^* \\
-C^*C & -A^*
\end{bmatrix}$$

are controllable through

$$\begin{bmatrix}
0 \\
-C^*
\end{bmatrix},$$

and observable through

$$\begin{bmatrix}
C & 0
\end{bmatrix}.$$  

Use the Popov-Bellman-Hautus (Kailath) test for controllability and observability.

(d) Hence, we have proven the statement: $G(j\omega)$ has a singular value equal to 1 if and only if

$$\begin{bmatrix}
A & BB^* \\
-C^*C & -A^*
\end{bmatrix}$$
has eigenvalue equal to $j\omega$. In other words: for all $\omega \in \mathbb{R}$, $G(j\omega)$ has no singular values equal to 1 if and only if
\[
\begin{bmatrix}
A & BB^* \\
-C^*C & -A^*
\end{bmatrix}
\]
has no imaginary axis eigenvalues. Generalize these two statements to the case where $G(j\omega)$ has a singular value equal to some positive number $\gamma \neq 1$. Verify this on the computer for a multivariable transfer function with several lightly damped, second order poles. Briefly explain your example.

(e) Prove the following: For $\gamma > 0$,
\[
\sup_{\omega \in \mathbb{R}} \bar{\sigma} [G(j\omega)] < \gamma
\]
if and only if
\[
\begin{bmatrix}
A & \frac{1}{\gamma}BB^* \\
-C^*C & -A^*
\end{bmatrix}
\]
has no imaginary axis eigenvalues.

Two good references for this problem are
