ME 234, MIMO Youla Parametrization Problems

1. Find an SISO plant that cannot be stabilized with constant-gain (proportional) feedback, but can be stabilized with dynamic feedback.

2. Consider a “plant” governed by the equations

\[
\begin{bmatrix}
\dot{x}(t) \\
e(t) \\
y(t)
\end{bmatrix} =
\begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & 0
\end{bmatrix}
\begin{bmatrix}
x(t) \\
d(t) \\
u(t)
\end{bmatrix}
\]

and a controller governed by

\[
\begin{bmatrix}
\dot{\eta}(t) \\
u(t)
\end{bmatrix} =
\begin{bmatrix}
A_c & B_c \\
C_c & D_c
\end{bmatrix}
\begin{bmatrix}
\eta(t) \\
y(t)
\end{bmatrix}
\]

Find a clean expression for the closed loop state-space matrix, i.e., the matrix \( M \) in the expression

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{\eta}(t) \\
e(t)
\end{bmatrix} =
M
\begin{bmatrix}
x(t) \\
\eta(t) \\
d(t)
\end{bmatrix}
\]

In your expression for \( M \), isolate the plant matrices from the controller matrices. Your answer should appear as \( R + UQV \), where the entries of \( R, U \) and \( V \) depend on the plant (though the dimensions of some zero and identity matrices depend on the controller’s dimension) and \( Q \) only depends on the controller.

3. Suppose \( \lambda \in \mathbb{C} \) with \( \text{Re}(\lambda) \geq 0 \), and \((A, B)\) is stabilizable and \((A, C)\) is detectable. Show that \( \lambda \) is a pole of \( G(s) := C (sI - A)^{-1} B \) if and only if \( \lambda \) is an eigenvalue of \( A \).

4. In this problem, we explore the stability of an interconnection of \( N \) linear systems, \( G_1, G_2, \ldots, G_N \). Suppose that \((A_i, B_i, C_i)\) is a stabilizable and detectable state-space realization of a multi-input/multi-output subsystem \( G_i \). Let \( x_i, u_i \) and \( y_i \) be the state, input and output of subsystem \( G_i \), so that

\[
\begin{align*}
\dot{x}_i(t) &= A_ix_i(t) + B_iu_i(t) \\
y_i(t) &= C_ix_i(t)
\end{align*}
\]

An interconnection of the subsystems is when the input to any subsystem is a linear combination of all of the outputs of the subsystems along with an additional external input \( v \)

\[
u_i(t) := v_i(t) + \sum_{j=1}^{N} K_{ij}y_j(t)\]
where the matrices \( \{K_{ij}\}_{i,j=1}^N \) are constant matrices (of appropriate dimension) called the interconnection matrices, and the \( v_i \) is called a junction input. Show that the interconnection (closed-loop system) is internally stable (ie., the closed-loop \( A \) matrix has all eigenvalues in the open-left-half plane) if and only if the transfer function matrix from \( v \) to \( y \) has all of its poles in the open-left-half plane. **Hint:** \((A, B)\) is stabilizable if and only if \((A + BF, B)\) is stabilizable for every \( F \).

5. For each of the realizations given below, determine if (independent of the \((A, B, C)\) data given) there are unobservable, or uncontrollable states, and if so, eliminate them to get a lower order realization

\[
\begin{bmatrix}
  A_{11} & 0 & 0 \\
  A_{21} & A_{22} & B_2 \\
  C_1 & C_2 & D 
\end{bmatrix},
\begin{bmatrix}
  A_{11} & 0 & B_1 \\
  A_{21} & A_{22} & 0 \\
  C_1 & C_2 & D 
\end{bmatrix}
\]

\[
\begin{bmatrix}
  A_{11} & 0 & B_1 \\
  A_{21} & A_{22} & B_2 \\
  C_1 & 0 & D 
\end{bmatrix},
\begin{bmatrix}
  A_{11} & 0 & B_1 \\
  A_{21} & A_{22} & B_2 \\
  0 & C_2 & D 
\end{bmatrix}
\]

\[
\begin{bmatrix}
  A_{11} & A_{12} & 0 \\
  0 & A_{22} & B_2 \\
  C_1 & C_2 & D 
\end{bmatrix},
\begin{bmatrix}
  A_{11} & A_{12} & B_1 \\
  0 & A_{22} & 0 \\
  C_1 & C_2 & D 
\end{bmatrix}
\]

\[
\begin{bmatrix}
  A_{11} & A_{12} & B_1 \\
  0 & A_{22} & B_2 \\
  C_1 & 0 & D 
\end{bmatrix},
\begin{bmatrix}
  A_{11} & A_{12} & B_1 \\
  0 & A_{22} & B_2 \\
  0 & C_2 & D 
\end{bmatrix}
\]

6. Using equations (5.9) and (5.10), verify that (5.11) is indeed a realization for \( T \).

7. Suppose the plant \( P \) is stable. Consider a 2 degree-of-freedom configuration as shown below.

Using the general parametrization we derived, show that all stabilizing 2-dof con-
trollers can be expressed as

\[ Q \text{ is any stable system, of dimension } n_u \times (n_r + n_y). \]

(a) Assume further that \( P \) is single-input, single-output (SISO). Denote the transfer function of \( Q \) as \( [Q_1(s) \ Q_2(s)] \). Write the closed-loop transfer functions, in terms of \( P \) and \( Q_i \) for the system shown below. Write the transfer functions in a \( 2 \times 3 \) matrix form, relating \([r; d; n]\) to \([u; y]\).

(b) Using Matlab (or Ptolemy), build your own tools for this (stable plant, 2-dof design, allow for MIMO plants), in m-files and/or Simulink.

(c) Suppose \( P = -0.3s + 1 \). Pick \( Q_2 \) so that \(|1 - PQ_2|\) is very small (compared to 1) in the frequency range \([0 1]\), and less than 2 for the entire frequency range. Pick \( Q_1 \) so that the controller actually processes \( r - y \), and not \( r \) and \( y \) individually. Do some simulations and/or frequency responses, and assess your design.

8. Suppose that \( A_{22} \) is Hurwitz, and a state-space model for a plant \( P \) is

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t) \\
y(t)
\end{bmatrix} =
\begin{bmatrix}
A_{11} & 0 & B_1 \\
A_{21} & A_{22} & B_2 \\
C_1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
u(t)
\end{bmatrix}
\]

Let the dimensions of \( x_1 \) and \( x_2 \) be \( n_1 \) and \( n_2 \) respectively. Find a parametrization of all stabilizing controllers for \( P \) of the form \( F_L(J, Q) \) where \( J \) has a \( n_1 \)th order state-space realization.

9. Consider the generalized MIMO “plant” in equation (??). Note that the 2, 2 element of \( D \) is defined to be 0. Suppose it is nonzero, using \( D_{22} \) to denote this matrix. Define new control inputs/sensed outputs as

\[
\begin{align*}
\tilde{u}(t) & := u(t) \\
\tilde{y}(t) & := y(t) - D_{22}u(t)
\end{align*}
\]

and let \( \tilde{P} \) be the system relating \( d \) and \( \tilde{u} \) to \( e \) and \( \tilde{y} \). Note that \( \tilde{P} \) has a identically zero “\( D_{22} \)” term. Parametrize all stabilizing controllers (processing \( \tilde{y} \), generating \( \tilde{u} \)) for \( \tilde{P} \). Explain how to “map” a stabilizing controller for \( \tilde{P} \) back into a stabilizing controller for \( P \). Are there any restrictions on \( D_Q \) so that various inverses exist?
10. The presentation in Section 2 concerned linear, time-invariant (LTI) plants and controllers. Everything goes through to the linear, time-varying (LTV) case as well. Suppose that $A : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$ is piecewise continuous, and bounded. Similar for $B(\cdot)$ and $C(\cdot)$.

**Definition:** Time-Varying Exponential Stability. Suppose $A : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$ is piecewise continuous, and bounded. Define $A$ (or more precisely, the linear system $\dot{x} = Ax$) to be exponentially stable if there exists a differentiable function $P : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$, constants $\gamma > 0$, and $0 < \alpha \leq \beta < \infty$ with $\alpha I \leq P(t) = P^T(t) \leq \beta I$ and

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) \leq -\gamma I$$

for all $t \geq 0$. This seems unnatural, as a definition should more directly associate with a single exponentially decaying bound on all of the solutions to the differential equation $\dot{x}(t) = A(t)x(t)$. In fact, these are equivalent, and the quadratic Lyapunov characterization of exponential stability, in terms of $P$, is correct (see Vidyasagar, *Nonlinear Systems Analysis*, 2nd Edition, Prentice Hall, 1993, pp. 202-204, for instance).

Verify these facts below:

(a) Suppose $0 < \alpha_i < \beta_i < \infty$ for $i = 1, 2$, and $X_i \in \mathbb{R}^{n_i \times n_i}$ with $X_i = X_i^T$. Further, assume that

$$-\beta_1 I \leq X_1 \leq -\alpha_1 I$$

$H \in \mathbb{R}^{n_1 \times n_2}$ is given. Show that there is an $\epsilon > 0$ such that

$$\begin{bmatrix} X_1 & \epsilon H \\ \epsilon H^T & \epsilon X_2 \end{bmatrix} < 0$$

(b) $A_i : \mathbb{R}_+ \rightarrow \mathbb{R}^{n_i \times n_i}$ are piecewise continuous, bounded and exponentially stable for $i = 1, 2$. Also suppose that $V : \mathbb{R}_+ \rightarrow \mathbb{R}^{n_2 \times n_1}$ is piecewise continuous and bounded. Then

$$\begin{bmatrix} A_1 & 0 \\ V & A_2 \end{bmatrix}$$

is exponentially stable.

(c) Suppose that $F(\cdot)$ is bounded, piecewise continuous function such that $A + BF$ is exponentially stable. Similarly, suppose that that $L(\cdot)$ is bounded, piecewise continuous function such that $A + LC$ is exponentially stable. Show that the controller involving $F$ and $L$ is exponentially stabilizing.

(d) In this part, use the $F$ and $L$ above. If $A_Q : \mathbb{R}_+ \rightarrow \mathbb{R}^{n_Q \times n_Q}$ is piecewise continuous, bounded and exponentially stable, $B_Q : \mathbb{R}_+ \rightarrow \mathbb{R}^{n_Q \times n_y}$, $C_Q : \mathbb{R}_+ \rightarrow \mathbb{R}^{n_u \times n_Q}$, $D_Q : \mathbb{R}_+ \rightarrow \mathbb{R}^{n_u \times n_y}$, all piecewise continuous and bounded, then the state-space formula in equation of all stabilizing controllers is an exponentially stabilizing LTV controller.
(e) Conversely, if $A_K(t), B_K(t), C_K(t), D_K(t)$ are the state-space matrices of a exponentially stabilizing LTV controller, with all entries bounded and piecewise continuous, then there is an exponentially stable LTV system $Q$ (with bounded, piecewise continuous state-space matrices $A_Q, B_Q, C_Q, D_Q$) such that the given formula is a realization of $A_K(t), B_K(t), C_K(t), D_K(t)$, with possibly some extra stable, uncontrollable and/or unobservable modes.

11. Improperly switching between stabilizing LTI controllers can lead to an unstable time-varying system. However, the controller architecture can always be chosen so that arbitrary switching is ok (at least from a stability point of view). Suppose that $K_1(s), K_2(s), \ldots, K_N(s)$ are each stabilizing controllers (each $K_i$ is finite dimensional, linear, time-invariant) for a linear, time-invariant $P$. Describe an implementation that has all of the LTI controllers, which is stabilizing for arbitrary switching among the controllers.